AMS212a HW1 Grading Rubric

Problem 1 (40 pts)

(a) – (4 pts) Substituting the transformation expression for \( T(x, t) \) into the PDE \( T_t - \kappa T_{xx} = 0 \) you should arrive at \( w_t - \kappa w_{xx} = 0 \).

– (3 pts) Substitute the transformation expression \( T(x = 0, t) \) evaluated at \( t = 0 \) into the equation \( T(0, t) - T_0 = 0 \).

– (3 pts) Substitute the transformation expression \( T(x = L, t) \) evaluated at \( t = 0 \) into the equation \( T(L, t) - T_1 = 0 \).

(b) – (2 pts) Pug in \( w(x, t) = X(x)T(t) \) into the PDE \( w_t - \kappa w_{xx} = 0 \) to get

\[
\begin{align*}
X'' + \lambda X &= 0 \\
T' + \kappa \lambda T &= 0
\end{align*}
\]

– (2 pts) Identify general solution for \( X(x) = a \sin(\sqrt{\lambda}x) + b \cos(\sqrt{\lambda}x) \)

– (2 pts) Applying the BC’s \( X(0) = X(L) = 0 \) show \( b = 0 \) and \( \lambda_n = (n\pi/L)^2 \)

– (2 pts) \( T_n(x) = e^{-\kappa(n\pi/L)^2t} \)

– (2 pts) \( w(x, t) = \sum_{n=1}^{\infty} w_n e^{-\kappa(n\pi/L)^2t} \sin(n\pi x/L) \)

– (3 pts) Find coefficients of \( w_n \) by multiplying both sides of \( w(x', 0) \) by \( \sin(m\pi x/L) \) for integer \( m \) and integrating over 0 to \( L \).

\[
w_n = \frac{2}{L} \int_0^L w(x', 0) \sin(m\pi x'/L) dx'
\]

– (3 pts) Series solution for \( w(x, t) \)

\[
w(x, t) = \sum_{n=1}^{\infty} \left( \frac{2}{L} \int_0^L (f(x') - \frac{1}{L}(T_1 - T_0)x' - T_0) \sin(n\pi x'/L) dx' \right) e^{-\kappa(n\pi/L)^2t} \sin(n\pi x/L)
\]

(c) – (10pts) As \( t \to \infty \), the exponential terms vanish yielding

\[
\lim_{t \to \infty} T(x, t) = \frac{1}{L}(T_1 - T_0)x + T_0
\]

Problem 2 (65 pts)

(a) (15pts) Substitute \( T(x, t) = F(x)G(t) \) into the PDE and arrive at

\[
\frac{G''}{A^2G} = \frac{(x^2 F')'}{F} = -\lambda
\]

(b) – (5 pts) For \( \lambda \neq 1/4 \) find that the general solution is \( F = ax^{\sigma_+} + bx^{\sigma_-} \) in the domain \( 1 < x < 2 \), where \( \sigma_{\pm} = \frac{-1 \pm \sqrt{1-4\lambda}}{2} \). One way to arrive at this is by plugging in \( x^\sigma \) into the ODE for \( F \) and solving for \( \sigma \).
- (5 pts) General solution for $\lambda = 1/4$ is $x^{-1/2}(c + d \ln x)$. One way to arrive at this is through reduction of order method by setting $F(x) = a(x)x^{-1/2}$ and plugging into the original ODE, where $\sigma = -1/2$ is the repeated root for $\lambda = 1/4$.

- (5 pts) Show that $\lambda = 1/4$ is not an eigenvalue and only leads to a trivial solution through the boundary conditions.

- (5 pts) Apply boundary conditions and find that $\sigma = -1/2 + \frac{2\pi n}{\ln 2}$ is the condition that solves the system of equations arrived at from each boundary condition at $x = 1$ and $x = 2$.

- (5 pts) Use the root equation $\sigma = -1/2 \pm \frac{\sqrt{1-4\lambda}}{2}$ to show $\sigma_+ - \sigma_- = \sqrt{1-4\lambda}$ and equating to the previous condition gives

$$4\lambda_n - 1 - (2\pi n/\ln 2)^2 = 0$$

for $n = 1, 2, \ldots$. 

- (5 pts) Find the corresponding eigenfunction by plugging in $\sigma(\lambda_n)$ to

$$F = a(x^{\sigma_+} - x^{\sigma_-}) = \alpha x^{-1/2} \sin(\pi n \ln x/\ln 2)$$

for $n = 1, 2, \ldots$ (BC gave $a = -b$)

(c) - (5 pts) Show S-L form

- (5 pts) Define general series solution as

$$T(x,t) = \sum_{n=1}^{\infty} T_n G_n(t) \frac{1}{\sqrt{x}} \sin \left( n\pi \ln x/\ln 2 \right)$$

- (5 pts) Find

$$G_n(t) = e^{-A^2 \lambda_n t}$$

- (5 pts) Find coefficient $T_n$ by applying the initial condition

$$T(x,0) = f(x) = \sum_{n=1}^{\infty} T_n \frac{1}{\sqrt{x}} \sin \left( n\pi \ln x/\ln 2 \right)$$

multiplying both sides by $\frac{1}{\sqrt{x}} \sin \left( m\pi \ln x'/\ln 2 \right)$, integrating from 1 to 2, and solving for $T_n$

$$T_n \frac{\ln 2}{2} = \int_1^2 f(x') \frac{1}{\sqrt{x}} \sin \left( m\pi \ln x'/\ln 2 \right) dx'$$