

AMS10 HW1

Problem 1. A vector space is a set V on which two operations, vector addition and scalar multiplication are defined. Construct an example with real vectors for each of the following conditions and show them to be true.

The vector addition operation $(+)$ must satisfy the following conditions:

- (1) Commutative law: For all vectors \vec{u} and \vec{v} in V , $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
- (2) Associative law: For all vectors \vec{u} , \vec{v} , and \vec{w} in V , $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$
- (3) Additive identity: The set V contains an additive identity element, denoted by a $\mathbf{0}$, such that for any vector \vec{v} in V , $\vec{u} + \mathbf{0} = \mathbf{0} + \vec{u}$
- (4) Additive inverses: For each vector \vec{v} in V , the equations $\vec{v} + \vec{x} = \mathbf{0}$ and $\vec{x} + \vec{v} = \mathbf{0}$ have a solution \vec{x} in V , called an additive inverse of \vec{v} , and denoted by $-\vec{v}$.

Note: *Closure*: If \vec{u} and \vec{v} are any vectors in V , then the sum $\vec{u} + \vec{v}$ belongs to V .

The scalar multiplication operation (\cdot) is defined between real numbers (or scalars) and vectors, and must satisfy the following conditions:

- (5) Distributive law: For all real numbers c and all vectors \vec{u}, \vec{v} in V , $c \cdot (\vec{u} + \vec{v}) = c \cdot \vec{u} + c \cdot \vec{v}$
- (6) Distributive law: For all real numbers c, d and all vectors \vec{v} in V , $(c + d) \cdot \vec{v} = c \cdot \vec{v} + d \cdot \vec{v}$
- (7) Associative law: For all real numbers c, d and all vectors \vec{v} in V , $c \cdot (d \cdot \vec{v}) = (cd) \cdot \vec{v}$
- (8) Unitary law: For all vectors \vec{v} in V , $1 \cdot \vec{v} = \vec{v}$

Note: *Closure*: If \vec{v} is any vector in V , and c is any real number, then the product $c \cdot \vec{v}$ belongs to V .

Problem 2. Draw the vector $\vec{a} - \vec{b}$ using the parallelogram law. Note that $\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$.

Problem 3. a. Calculate the absolute value $|z|$ for the following complex numbers

i. $z = \frac{1}{2} + i\frac{\sqrt{3}}{2}$

ii. $z = 5 \cdot \frac{-\sqrt{2}}{2} + i5 \cdot \frac{\sqrt{2}}{2}$

b. Write down the complex exponential form of

i. $z = \frac{1}{2} + i\frac{\sqrt{3}}{2}$

ii. $z = 5 \cdot \frac{-\sqrt{2}}{2} + i5 \cdot \frac{\sqrt{2}}{2}$

(Hint: there are an infinite number of representations)

c. Use the complex exponential form of $z = 10\frac{\sqrt{3}}{2} + i10\frac{1}{2}$ to show that $x^2 = 10\frac{\sqrt{3}}{2} + i10\frac{1}{2}$ has only two distinct solutions.

Problem 4 Simplifying fractions of complex numbers. Suppose we have a complex number expressed as the division of two distinct complex numbers of the form

$$z = \frac{a_1 + ib_1}{a_2 + ib_2}$$

Then a method of finding the real and imaginary component involves the complex conjugate of the denominator in order to remove any imaginary numbers from the denominator. Recall that $z\bar{z} = |z|^2$ is a real non-negative number. The following is an illustration of this

$$z = \frac{a_1 + ib_1}{a_2 + ib_2} \tag{1}$$

$$= \frac{1}{1} \cdot \frac{a_1 + ib_1}{a_2 + ib_2} \tag{2}$$

$$= \frac{a_2 - ib_2}{a_2 - ib_2} \cdot \frac{a_1 + ib_1}{a_2 + ib_2} \tag{3}$$

$$= \frac{(a_2 - ib_2)(a_1 + ib_1)}{a_2^2 + b_2^2}. \tag{4}$$

Apply this method to find the real and imaginary components of the following expressions (don't just plug in values but show work):

$$\text{a. } \frac{3 + i}{2 + i} \tag{5}$$

$$\text{b. } \frac{1 + i}{2 - i} \tag{6}$$

$$\text{c. } \frac{1 - 2i}{2 + 2i} \tag{7}$$

$$\text{d. } \frac{-1 - 2i}{1 + 2i} \tag{8}$$

Problem 5 Division of complex numbers using the exponential form. Suppose you are given two complex numbers in the form $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then the division is given by

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

which is equivalent to

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} (\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)).$$

a. Find the argument (or phase) for the numerator and denominator of (c) and (d) in Problem 4. You can find the argument using arctan and applying the appropriate adjustment. Show work. Verify the answer using the Matlab command **angle** (e.g. `angle(1+i)`).

b. Find the modulus (or absolute value) for the numerator and denominator of (c) and (d) in Problem 4. You may work out by hand or use the Matlab command **abs** (e.g. `abs(1+i)`).

c. Using the results from (a) and (b), write the complex exponential form for the numerator and denominator of (c) and (d) in Problem 4. Calculate the real and imaginary component of (c) and (d) by dividing the exponential functions. Verify that the final complex exponential form ($z_1/z_2 = re^{i\theta} = \cos(\theta) + i\sin(\theta)$) is equivalent to the results found in Problem 4. Recall, two complex numbers $a + ib$ and $c + id$ are equal if and only if $a = c$ and $b = d$.

Problem 6. Given the complex number $z = -1 + 2i$ find the m distinct m -th roots ($z^{1/m}$) for

- a. $m=10$
- b. $m=45$
- c. $m=100$

It is ok to leave in exponential form.

Problem 7. Calculate the following powers of $z = -1 + 2i$

- a. z^{10}
- b. z^{15}
- c. z^{200}

It is ok to leave in exponential form.

Problem 8 Matlab: Construct the following vectors in Matlab:

- a. A 2x1 vector with all 1's in the entries and define it u .
- b. A 2x1 vector with the first element 4 and the second element 3 and define it v
- c. What is $5u+v$? Verify using Matlab

*No need to show any proof of using Matlab.