AMS10 HW1 Grading Rubric

Problem 1 (16pts- 2pts/each). ***Left hand side is shown to equal right hand side using examples with real vectors.*** A vector space is a set V on which two operations, vector addition and scalar multiplication are defined. Construct an example with real vectors for each of the following conditions and show them to be true.

The vector addition operation (+) must satisfy the following conditions:

(1) Commutative law: For all vectors $\vec{u}$ and $\vec{v}$ in V, $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

(2) Associative law: For all vectors $\vec{u}$,$\vec{v}$, and $\vec{w}$ in V, $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$

(3) Additive identity: The set V contains an additive identity element, denoted by a $\mathbf{0}$, such that for any vector $\vec{v}$ in V, $\vec{u} + \mathbf{0} = \mathbf{0} + \vec{u}$

(4) Additive inverses: For each vector $\vec{v}$ in V, the equations $\vec{v} + \vec{x} = \mathbf{0}$ and $\vec{x} + \vec{v} = \mathbf{0}$ have a solution $\vec{x}$ in V, called an additive inverse of $\vec{v}$, and denoted by $-\vec{v}$.

Note: *Closure*: If $\vec{u}$ and $\vec{v}$ are any vectors in V, then the sum $\vec{u} + \vec{v}$ belongs to V.

The scalar multiplication operation $(\cdot)$ is defined between real numbers (or scalars) and vectors, and must satisfy the following conditions:

(5) Distributive law: For all real numbers c and all vectors $\vec{u}, \vec{v}$ in V, $c \cdot (\vec{u} + \vec{v}) = c \cdot \vec{u} + c \cdot \vec{v}$

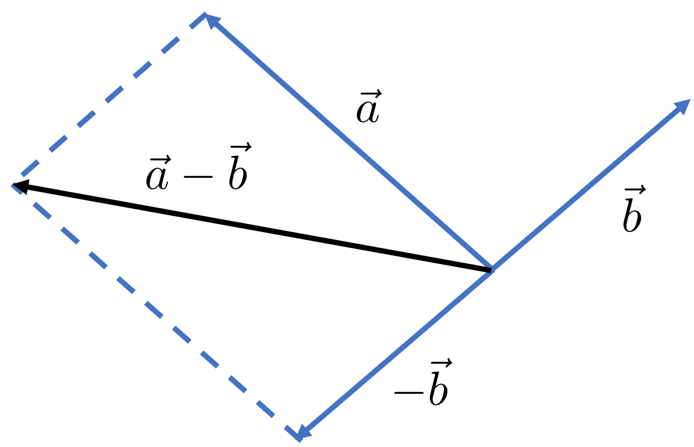
(6) Distributive law: For all real numbers c, d and all vectors $\vec{v}$ in V, $(c + d) \cdot \vec{v} = c \cdot \vec{v} + d \cdot \vec{v}$

(7) Associative law: For all real numbers c, d and all vectors $\vec{v}$ in V, $c \cdot (d \cdot \vec{v}) = (cd) \cdot \vec{v}$

(8) Unitary law: For all vectors $\vec{v}$ in V, $1 \cdot \vec{v} = \vec{v}$

Note: *Closure*: If $\vec{v}$ is any vector in V, and c is any real number, then the product $c \cdot \vec{v}$ belongs to V.

Problem 2 (5pts). Draw the vector $\vec{a} - \vec{b}$ using the parallelogram law. Note that $\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$,



$\vec{a}$
$\vec{a}-\vec{b}$
$\vec{b}$
$-\vec{b}$

Problem 3 (18 pts). a.**(4pts - 2pts each)** Calculate the absolute value $|z|$ for the following complex numbers

i. $z = \frac{1}{2} + i\frac{\sqrt{3}}{2}$

ii. $z = 5 \cdot \frac{-\sqrt{2}}{2} + i5 \cdot \frac{\sqrt{2}}{2}$

Solution:

i. $|z| = 1$

ii. $|z| = 5$

b. **(4pts - 2pts each)** Write down the complex exponential form of

i. $z = \frac{1}{2} + i\frac{\sqrt{3}}{2}$

ii. $z = 5 \cdot \frac{-\sqrt{2}}{2} + i5 \cdot \frac{\sqrt{2}}{2}$

(Hint: there are an infinite number of representations)
Solution:

i. $z = e^{i(\frac{\pi}{3}+2\pi k)}$ for $k = 0, \pm 1, \pm 2, \ldots$

ii. $z = 5e^{i(\frac{3\pi}{4}+2\pi k)}$ for $k = 0, \pm 1, \pm 2, \ldots$

c. **(10 pts)**Use the complex exponential form of $z = 10\frac{\sqrt{3}}{2} + i10\frac{1}{2}$ to show that $x^2 = 10\frac{\sqrt{3}}{2} + i10\frac{1}{2}$ has only two distinct solutions.

We have

$$z^2 = 10\frac{\sqrt{3}}{2} + i10\frac{1}{2} \tag{11}$$

$$z^2 = 10e^{i(\frac{\pi}{6}+2\pi k)}(\text{10pts if you got this far}) \tag{12}$$

$$(z^2)^{1/2} = (10e^{i(\frac{\pi}{6}+2\pi k)})^{1/2} \tag{13}$$

$$z = \sqrt{10}e^{i(\frac{\pi}{12}+\pi k)} \tag{14}$$

for $k = 0, \pm 1, \pm 2, \ldots$

Note that

$$\sqrt{10}e^{i(\frac{\pi}{12}+\pi)} = \sqrt{10}e^{i(\frac{\pi}{12}+3\pi)} = \sqrt{10}e^{i(\frac{\pi}{12}+5\pi)} = \sqrt{10}e^{i(\frac{\pi}{12}+(2k+1)\pi)}$$

and

$$\sqrt{10}e^{i(\frac{\pi}{12}+2\pi)} = \sqrt{10}e^{i(\frac{\pi}{12}+4\pi)} = \sqrt{10}e^{i(\frac{\pi}{12}+6\pi)} = \sqrt{10}e^{i(\frac{\pi}{12}+2k\pi)}$$

and so there are only two unique solutions (full credit).

Problem 4 (15pts-3pts/each). Simplifying fractions of complex numbers. Suppose we have a complex number expressed as the division of two distinct complex numbers of the form

$$z = \frac{a_1 + ib_1}{a_2 + ib_2}.$$

Then a method of finding the real and imaginary component involves the complex conjugate of the denominator in order to remove any imaginary numbers from the denominator. Recall that $z\bar{z} = |z|^2$ is a real non-negative number. The following is an illustration of this

$$\begin{aligned} z &= \frac{a_1 + ib_1}{a_2 + ib_2} && (15) \\ &= \frac{1}{1} \cdot \frac{a_1 + ib_1}{a_2 + ib_2} && (16) \\ &= \frac{a_2 - ib_2}{a_2 - ib_2} \cdot \frac{a_1 + ib_1}{a_2 + ib_2} && (17) \\ &= \frac{(a_2 - ib_2)(a_1 + ib_1)}{a_2^2 + b_2^2}. && (18) \end{aligned}$$

Apply this method to find the real and imaginary components of the following expressions (don't just plug in values but show work):

$$\begin{aligned} &\text{a. } \frac{3+i}{-2+i} && (19) \\ &\text{b. } \frac{1+i}{2-i} && (20) \\ &\text{c. } \frac{1-2i}{2+2i} && (21) \\ &\text{d. } \frac{-1-2i}{1+2i} && (22) \\ & && (23) \end{aligned}$$

Solution:

$$\begin{aligned} &\text{a. } \frac{3+i}{2+i} = \frac{2-i}{2-i} \cdot \frac{3+i}{2+i} = \frac{6+2i-3i+1}{4+1} = \frac{7-i}{5} = \frac{7}{5} - \frac{1}{5}i && (24) \\ &\text{b. } \frac{1+i}{2-i} = \frac{2+i}{2+i} \cdot \frac{1+i}{2-i} = \frac{2+3i-1}{4+1} = \frac{1+3i}{5} = \frac{1}{5} + \frac{3}{5}i && (25) \\ &\text{c. } \frac{1-2i}{2+2i} = \frac{2-2i}{2-2i} \cdot \frac{1-2i}{2+2i} = \frac{2-4i-2i-4}{4+4} = \frac{-2-6i}{8} = -\frac{1}{4} - \frac{3}{4}i && (26) \\ &\text{d. } \frac{-1-2i}{1+2i} = \frac{1-2i}{1-2i} \cdot \frac{-1-2i}{1+2i} = \frac{-1+2i-2i-4}{4+1} = \frac{-5}{5} = -1 && (27) \\ & && (28) \end{aligned}$$

Problem 5 (18 pts- 3pts/each). Division of complex numbers using the exponential form. Suppose you are given two complex numbers in the form $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then the division is given by

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \frac{e^{i\theta_1}}{e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

which is equivalent to

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \left(\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2) \right).$$

i. Find the argument (or phase) for the numerator and denominator of (c) and (d) in problem 4. Find the angle using arctan and applying the appropriate adjustment. Show work. Verify the answer using the Matlab command **angle** (e.g. angle(1+i)).

Solution:

$$\begin{aligned}
\text{c. } \arg(1-2i) &= \arctan(-2) && (29)\\
\arg(2+2i) &= \arctan(1) && (30)\\
\text{d. } \arg(-1-2i) &= \arctan(2)-\pi && (31)\\
\arg(1+2i) &= \arctan(2) && (32)
\end{aligned}$$

ii. Find the modulus (or absolute value) for the numerator and denominator of (c) and (d) in Problem 4. You may work out by hand or use the Matlab command **abs** (e.g. abs(1+i)).

Solution:

$$\begin{aligned}
\text{c. } |1-2i| &= \sqrt{5} && (33)\\
|2+2i| &= \sqrt{8} && (34)\\
\text{d. } |-1-2i| &= \sqrt{5} && (35)\\
|1+2i| &= \sqrt{5} && (36)
\end{aligned}$$

iii. Using the results from (a) and (b), write the complex exponential form for the numerator and denominator of (c) and (d) in Problem 4. Calculate the real and imaginary component of (c) and (d) by dividing the complex exponential functions. Verify that the final complex exponential form ($z_1/z_2 = re^{i\theta} = \cos(\theta) + i\sin(\theta)$) is equivalent to the results found in Problem 4. Recall, two complex numbers $a+ib$ and $c+id$ are equal if and only if $a=c$ and $b=d$.

Solution: Applying the definition of the complex exponential we find that the real part is

$$\begin{aligned}
\text{c. Re: } \frac{r_1}{r_2}\cos(\theta_1-\theta_2) &= \frac{\sqrt{5}}{\sqrt{8}}\cos(\arctan(-2)-\arctan(1)) = -.25 && (37)\\
\text{Im: } \frac{r_1}{r_2}\cos(\theta_1-\theta_2) &= \frac{\sqrt{5}}{\sqrt{8}}\sin(\arctan(-2)-\arctan(1)) = -.75 && (38)\\
\text{d. Re: } \frac{r_1}{r_2}\cos(\theta_1-\theta_2) &= \cos(-\pi) = -1 && (39)\\
\text{Im: } \frac{r_1}{r_2}\cos(\theta_1-\theta_2) &= \sin(-\pi) = 0 && (40)
\end{aligned}$$

Problem 6 (9pts-3pts/each). Given the complex number $z = -1+2i$ find the m distinct m-th roots ($z^{1/m}$) for

a. m=10

b. m=45

c. m=100

It is ok to leave in exponential form.

Solution: The complex form is

$$z = \sqrt{5}e^{i\theta}$$

where

$$\theta = \arctan(-2) + \pi.$$

$$\text{a. } 5^{1/20}e^{i(\theta+2\pi k)/10} \text{ for } k = 0, 1, \ldots, 9 \tag{41}$$

$$\text{b. } 5^{1/90}e^{i(\theta+2\pi k)/45} \text{ for } k = 0, 1, \ldots, 44 \tag{42}$$

$$\text{c. } 5^{1/200}e^{i(\theta+2\pi k)/100} \text{ for } k = 0, 1, \ldots, 99 \tag{43}$$

$$\tag{44}$$

Problem 7 (9pts-3pts/each). Calculate the following powers of $z = -1 + 2i$

a. z^{10}
b. z^{15}
c. z^{200}
It is ok to leave in exponential form.
Solution

$$\text{a. } 5^5 e^{i\theta 10} \tag{45}$$

$$\text{b. } 5^{15/2} e^{i\theta 15} \tag{46}$$

$$\text{c. } 5^{100} e^{i\theta 200} \tag{47}$$

$$\tag{48}$$

Problem 8. Construct the following vectors in Matlab:

a. A 2x1 vector with all 1's in the entries and define it u.

b. A 2x1 vector with the first element 4 and the second element 3 and define it v

c. What is 5u+v? Verify using Matlab

Write down using Matlab syntax the exact expression you would type into the command window for a-c.

Solution:
a. u=[1;1];
b. v=[4;3];
c. 5*u+v , answer: 5u+v=[9;8]

***13pts for submission**