AMS10 HW2 Grading Rubric

Problem 1.Simplifying fractions of complex numbers. Suppose we have a complex number expressed as the division of two distinct complex numbers of the form

$$z = \frac{a_1 + ib_1}{a_2 + ib_2}.$$

Then a method of finding the real and imaginary component involves the complex conjugate of the denominator in order to remove any imaginary numbers from the denominator. Recall that $z\bar{z} = |z|^2$ is a real non-negative number. The following is an illustration of this

$$\begin{aligned} z &= \frac{a_1 + ib_1}{a_2 + ib_2} && (18) \\ &= \frac{1}{1} \cdot \frac{a_1 + ib_1}{a_2 + ib_2} && (19) \\ &= \frac{a_2 - ib_2}{a_2 - ib_2} \cdot \frac{a_1 + ib_1}{a_2 + ib_2} && (20) \\ &= \frac{(a_2 - ib_2)(a_1 + ib_1)}{a_2^2 + b_2^2}. && (21) \end{aligned}$$

Apply this method to find the real and imaginary components of the following expressions (don't just plug in values but show work):

$$\text{a. } \frac{3+i}{-2+i} \quad (22)$$

$$\text{b. } \frac{1+i}{2-i} \quad (23)$$

$$\text{c. } \frac{1-2i}{2+2i} \quad (24)$$

$$\text{d. } \frac{-1-2i}{1+2i} \quad (25)$$

$$\text{e. } \frac{-1+2i}{5+3i} \quad (26)$$

(27)

Solution:

$$\text{a. } \frac{3+i}{2+i} = \frac{2-i}{2-i} \cdot \frac{3+i}{2+i} = \frac{6+2i-3i+1}{4+1} = \frac{7-i}{5} = \frac{7}{5} - \frac{1}{5}i \quad (28)$$

$$\text{b. } \frac{1+i}{2-i} = \frac{2+i}{2+i} \cdot \frac{1+i}{2-i} = \frac{2+3i-1}{4+1} = \frac{1+3i}{5} = \frac{1}{5} + \frac{3}{5}i \quad (29)$$

$$(5pts)\text{c. } \frac{1-2i}{2+2i} = \frac{2-2i}{2-2i} \cdot \frac{1-2i}{2+2i} = \frac{2-4i-2i-4}{4+4} = \frac{-2-6i}{8} = -\frac{1}{4} - \frac{3}{4}i \quad (30)$$

$$(5pts)\text{d. } \frac{-1-2i}{1+2i} = \frac{1-2i}{1-2i} \cdot \frac{-1-2i}{1+2i} = \frac{-1+2i-2i-4}{4+1} = \frac{-5}{5} = -1 \quad (31)$$

$$(5pts)\text{e. } \frac{-1+2i}{5+3i} = \frac{5-3i}{5-3i} \cdot \frac{-1+2i}{5+3i} = \frac{-5+3i+10i+6}{25+9} = \frac{1+13i}{34} = \frac{1}{34} + \frac{13}{34}i \quad (32)$$

Problem 2. Division of complex numbers using the exponential form. Suppose you are given two complex numbers in the form $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then the division is given by

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \frac{e^{i\theta_1}}{e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

which is equivalent to

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right).$$

i. Find the argument (or phase) for the numerator and denominator of (a)-(e) in problem 1. Find the angle using arctan and applying the appropriate adjustment. Show work. Verify the answer using the Matlab command **angle** (e.g. angle(1+i)).

Solution:

$$\begin{aligned}
\text{a. } \arg(3+i) &= \arctan(1/3) && (33)\\
\arg(2+i) &= \arctan(1/2) && (34)\\
\text{b. } \arg(1+i) &= \arctan(1) && (35)\\
\arg(2-i) &= \arctan(-1/2) && (36)\\
(5pts)\text{c. } \arg(1-2i) &= \arctan(-2) && (37)\\
\arg(2+2i) &= \arctan(1) && (38)\\
(5pts)\text{d. } \arg(-1-2i) &= \arctan(2) - \pi && (39)\\
\arg(1+2i) &= \arctan(2) && (40)\\
(5pts)\text{e. } \arg(-1+2i) &= \arctan(-2) + \pi && (41)\\
\arg(5+3i) &= \arctan(3/5) && (42)\\
& && (43)
\end{aligned}$$

ii. Find the modulus (or absolute value) for the numerator and denominator of (a)-(e) in Problem 1. You may work out by hand or use the Matlab command **abs** (e.g. abs(1+i)). You can also work out by hand and verify using Matlab.

Solution:

$$\begin{aligned}
\text{a. } |3+i| &= \sqrt{3^2+1^2} = \sqrt{10} && (44)\\
|2+i| &= \sqrt{(2)^2+1^2} = \sqrt{5} && (45)\\
\text{b. } |1+i| &= \sqrt{2} && (46)\\
|2-i| &= \sqrt{5} && (47)\\
(5pts)\text{c. } |1-2i| &= \sqrt{5} && (48)\\
|2+2i| &= \sqrt{8} && (49)\\
(5pts)\text{d. } |-1-2i| &= \sqrt{5} && (50)\\
|1+2i| &= \sqrt{5} && (51)\\
(5pts)\text{e. } |-1+2i| &= \sqrt{5} && (52)\\
|5+3i| &= \sqrt{34} && (53)\\
& && (54)
\end{aligned}$$

iii. Calculate the real and imaginary component of (a)-(e) using the complex exponential form and verify that it is equal to the results found in Problem 1. Recall, two complex numbers $a + ib$ and $c + id$ are equal if and only if $a = c$ and $b = d$.

Solution: Applying the definition of the complex exponential we find that the real part is

$$\text{a. Re: } \frac{r_1}{r_2}\cos(\theta_1-\theta_2) = \frac{\sqrt{10}}{\sqrt{5}}\cos(\arctan(1/3)-\arctan(1/2)) = 1.4 \quad (55)$$

$$\text{Im: } \frac{r_1}{r_2}\cos(\theta_1-\theta_2) = \frac{\sqrt{10}}{\sqrt{5}}\sin(\arctan(1/3)-\arctan(1/2)) = -.2 \quad (56)$$

$$\text{b. Re: } \frac{r_1}{r_2}\cos(\theta_1-\theta_2) = \frac{\sqrt{2}}{\sqrt{5}}\cos(\arctan(1)-\arctan(-1/2)) = .2 \quad (57)$$

$$\text{Im: } \frac{r_1}{r_2}\cos(\theta_1-\theta_2) = \frac{\sqrt{2}}{\sqrt{5}}\sin(\arctan(1)-\arctan(-1/2)) = .6 \quad (58)$$

$$(5pts)\text{c. Re: } \frac{r_1}{r_2}\cos(\theta_1-\theta_2) = \frac{\sqrt{5}}{\sqrt{8}}\cos(\arctan(-2)-\arctan(1)) = -.25 \quad (59)$$

$$\text{Im: } \frac{r_1}{r_2}\cos(\theta_1-\theta_2) = \frac{\sqrt{5}}{\sqrt{8}}\sin(\arctan(-2)-\arctan(1)) = -.75 \quad (60)$$

$$(5pts)\text{d. Re: } \frac{r_1}{r_2}\cos(\theta_1-\theta_2) = \cos(-\pi) = -1 \quad (61)$$

$$\text{Im: } \frac{r_1}{r_2}\cos(\theta_1-\theta_2) = \sin(-\pi) = 0 \quad (62)$$

$$(5pts)\text{e. Re: } \frac{r_1}{r_2}\cos(\theta_1-\theta_2) = \frac{\sqrt{5}}{\sqrt{34}}\cos(\arctan(-2)+\pi-\arctan(3/5)) = .0294 \quad (63)$$

$$\text{Im: } \frac{r_1}{r_2}\cos(\theta_1-\theta_2) = \frac{\sqrt{5}}{\sqrt{34}}\sin(\arctan(-2)+\pi-\arctan(3/5)) = 0.3824i \quad (64)$$

(65)

Problem 3. Given the complex number $z = -1 + 2i$ find the m distinct m-th roots ($z^{1/m}$) for

a. m=10

b. m=45

c. m=100

It is ok to leave in exponential form.

Solution: The complex form is

$$z = \sqrt{5}e^{i\theta}$$

where

$$\theta = \arctan(-2) + \pi.$$

$$(5pts)\text{a. } 5^{1/20}e^{i(\theta+2\pi k)/10} \text{ for } k = 0, 1, \ldots, 9 \quad (66)$$

$$(5pts)\text{b. } 5^{1/90}e^{i(\theta+2\pi k)/45} \text{ for } k = 0, 1, \ldots, 44 \quad (67)$$

$$(5pts)\text{c. } 5^{1/200}e^{i(\theta+2\pi k)/100} \text{ for } k = 0, 1, \ldots, 99 \quad (68)$$

(69)

Problem 4.Calculate the following powers of $z = -1 + 2i$

a. z^{10}

b. z^{15}

c. z^{200}

It is ok to leave in exponential form.

Solution

$$\text{a. } 5^5 e^{i\theta 10} \tag{70}$$
$$\text{b. } 5^{15/2} e^{i\theta 15} \tag{71}$$
$$\text{c. } 5^{100} e^{i\theta 200} \tag{72}$$
$$\tag{73}$$

Problem 5. Consider the following linear system of equations

$$x_1 + 5x_2 + 2x_3 = 3 \tag{74}$$
$$x_2 + 5x_3 = 5 \tag{75}$$
$$2x_1 + x_2 + x_3 = 1. \tag{76}$$

a. Apply the following sequence of elementary operations

1. $L_2 \leftrightarrow L_3$
2. $-2L_1 + L_2 \rightarrow L_2$
3. $\frac{1}{9}L_2 + L_3 \rightarrow L_3$.

You should end up in echelon form (in fact you will have triangular form)
Solution
1. $L_2 \leftrightarrow L_3$

$$x_1 + 5x_2 + 2x_3 = 3 \tag{77}$$
$$2x_1 + x_2 + x_3 = 1 \tag{78}$$
$$x_2 + 5x_3 = 5 \tag{79}$$

2. $-2L_1 + L_2 \rightarrow L_2$

$$x_1 + 5x_2 + 2x_3 = 3 \tag{80}$$
$$0 - 9x_2 - 3x_3 = -5 \tag{81}$$
$$x_2 + 5x_3 = 5 \tag{82}$$

3. $\frac{1}{9}L_2 + L_3 \rightarrow L_3$

$$x_1 + 5x_2 + 2x_3 = 3 \tag{83}$$
$$0 - 9x_2 - 3x_3 = -5 \tag{84}$$
$$0 + (-1/3 + 5)x_3 = 5 - 5/9 \tag{85}$$

b.(15pts faithful effort - 5pts for each variable) Solve for x_1, x_2, and x_3.

$$x_3 = 20/21 = .9524$$
$$x_2 = \frac{-1}{9}(-5 + 20/7) = .2381$$
$$x_3 = 3 - 40/21 + \frac{5}{9}(-5 + 20/7) = -.0952$$

c. (10pts faithful effort) Does this system have no solution, one solution, or infinite solutions. How can you be sure? (hint: apply theorem for systems in echelon form)

This system has one unique solution. It has two equations and two unknowns in echelon form and so it has a unique solution.