AMS212a HW2

1. (45 pts) Let \( u(x, y) \) and \( v(r, \theta) \) be two functions which, for \( x = r \cos(\theta) \) and \( y = r \sin(\theta) \), satisfy

\[
  u(x, y) = v(r, \theta) \tag{2}
\]

(a) (15 pts) Show that, for \( x = r \cos(\theta) \) and \( y = r \sin(\theta) \) we have

\[
  u_x(x, y) = \cos(\theta)v_r(r, \theta) - \frac{\sin(\theta)}{r}v_\theta(r, \theta) \tag{3}
\]

and

\[
  u_y(x, y) = \sin(\theta)v_r(r, \theta) - \frac{\cos(\theta)}{r}v_\theta(r, \theta) \tag{4}
\]

[Method: Differentiate the identity \( u(r \cos(\theta), r \sin(\theta)) = v(r, \theta) \) with respect to \( r \) and \( \theta \) and solve for \( u_x \) and \( u_y \)]

(b) (15 pts) Noting that equations (3) and (4) are of the form (2), apply the method above to (3) and (4) to obtain explicit expressions for \( u_{xx}(x, y) \) and \( u_{yy}(x, y) \) for \( x = r \cos(\theta) \) and \( y = r \sin(\theta) \).

(c) (15 pts) Show that in two spatial dimensions, for \( x = r \cos(\theta) \) and \( y = r \sin(\theta) \) we have

\[
  \nabla^2 u(x, y) = u_{xx}(x, y) + u_{yy}(x, y) = v_{rr}(r, \theta) + \frac{1}{r}v_r(r, \theta) + \frac{1}{r^2}v_{\theta\theta}(r, \theta)
\]

Note, also, that

\[
  v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\theta\theta} = \frac{1}{r}(rv_r)_r + \frac{1}{r^2}v_{\theta\theta}.
\]

2. (55 pts)

(a) (25 pts) Solve \( u_{xx} + u_{yy} = 0 \) in the exterior \( \{ r > a \} \) of a disk, with the boundary condition \( u = 1 + 3 \sin(\theta) \) on \( r = a \) and the condition that \( u \) be bounded at infinity.

(b) (30 pts) Solve \( u_{xx} + u_{yy} = 0 \) in the disk \( r < a \) with the boundary condition

\[
  u_r - hu = f(\theta)
\]

where \( f \) is an arbitrary function. Express the answer in terms of the Fourier coefficients of \( f \).

Hint: Should find coefficient of Fourier expansion to include \( c_n(r) = Ar^n \) for \( n \neq 0 \) for part (a) and \( c_n(r) = A_n r^n + B_n r^{-n} \) for \( n \neq 0 \) for part (b) and don’t forget the \( n = 0 \) case. Show ALL work. Final solution should include 3 cases, one of which does not have a solution—look for cases of singularities when solving coefficients.