AMS10 HW2 Solutions

Problem 1 **(60 pts- 15pts for i-iv)–2pts for each missing subsection a-g (parts h and i not graded)**.

i. Consider the following system of linear equations

$$4x_1 + 2x_2 + x_3 = 1 \tag{13}$$
$$x_1 + x_2 + 2x_3 = 2 \tag{14}$$
$$2x_1 + 2x_2 + x_3 = 5 \tag{15}$$

a. Put into augmented matrix form

$$\left(\begin{array}{ccc|c} 4 & 2 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 2 & 2 & 1 & 5 \end{array}\right) \tag{16}$$

b. Put in echelon form

$$(-1/4)L_1 + L_2 \to L_2 \implies \left(\begin{array}{ccc|c} 4 & 2 & 1 & 1 \\ 0 & 1/2 & 7/4 & 7/4 \\ 2 & 2 & 1 & 5 \end{array}\right) \tag{17}$$

$$(-1/2)L_1 + L_3 \to L_3 \implies \left(\begin{array}{ccc|c} 4 & 2 & 1 & 1 \\ 0 & 1/2 & 7/4 & 7/4 \\ 0 & 1 & 1/2 & 5 - 1/2 \end{array}\right) \tag{18}$$

$$(-2)L_2 + L_3 \to L_3 \implies \left(\begin{array}{ccc|c} 4 & 2 & 1 & 1 \\ 0 & 1/2 & 7/4 & 7/4 \\ 0 & 0 & -3 & 1 \end{array}\right) \tag{19}$$

c. solve for x_3, x_2, x_1 in that order

$$-3x_3 = 1 \implies x_3 = -1/3$$

$$(1/2)x_2 = (7/4)(1 - x_3) = 7/4 + 7/12 \implies x_2 = 7/2 + 7/6$$

$$4x_1 = 1 - x_3 - 2x_2 = 1 + 1/3 - 7 - 7/3 = -8 \implies x_1 = -2$$

d. How many pivot points, hence, pivot variables? 3

e. How many free variables are there? 0

f. How many solutions are there if any? one solution

h. Put into canonical form. Note, pivot points should be the only non-zero number in its column.

$$\begin{array}{c} (1/4)L_1 \to L_4 \\ 2L_2 \to L_2 \\ (-1/3)L_3 \to L_3 \end{array} \implies \left(\begin{array}{ccc|c} 1 & 1/2 & 1/4 & 1/4 \\ 0 & 1 & 7/2 & 7/2 \\ 0 & 0 & 1 & -1/3 \end{array}\right) \tag{20}$$

$$(-7/2)L_3 + L_2 \to L_2 \implies \left(\begin{array}{ccc|c} 1 & 1/2 & 1/4 & 1/4 \\ 0 & 1 & 0 & 7/2 + 7/6 \\ 0 & 0 & 1 & -1/3 \end{array}\right) \tag{21}$$

$$(-1/4)L_3 + L_1 \to L_1 \implies \left(\begin{array}{ccc|c} 1 & 1/2 & 0 & 1/4 + 1/12 \\ 0 & 1 & 0 & 7/2 + 7/6 \\ 0 & 0 & 1 & -1/3 \end{array}\right) \tag{22}$$

$$(-1/2)L_2 + L_1 \to L_1 \implies \left(\begin{array}{ccc:c} 1 & 0 & 0 & -6/4 - 6/12 \\ 0 & 1 & 0 & 7/2 + 7/6 \\ 0 & 0 & 1 & -1/3 \end{array}\right) \tag{23}$$

i. Put final solution in decimal form and compare with solution given by Matlab by applying the command **rref** to the original augmented matrix.

$$\mathrm{rref}\left([4,2,1,1;1,1,2,2;2,2,1,5]\right) = \left(\begin{array}{ccc:c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 4.\bar{6} \\ 0 & 0 & 1 & -.\bar{3} \end{array}\right) \tag{24}$$

j. In Matlab, take an augmented matrix from one of the intermediate steps and apply rref. Is the solution different? No.

Problem 1. ii. Consider the following system of linear equations

$$4x_1 + 8x_2 + x_3 = 1 \tag{25}$$
$$x_1 + 2x_2 + 2x_3 = 1 \tag{26}$$
$$x_1 + x_2 + 2x_3 = 2 \tag{27}$$

(10pts) a. Put into augmented matrix form

$$\left(\begin{array}{ccc:c} 4 & 8 & 1 & 1 \\ 1 & 2 & 2 & 1 \\ 1 & 1 & 2 & 2 \end{array}\right) \tag{28}$$

(10pts) b. Put in echelon form

$$(-1/4)L_1 + L_2 \to L_2 \implies \left(\begin{array}{ccc:c} 4 & 8 & 1 & 1 \\ 0 & 0 & 7/4 & 3/4 \\ 1 & 1 & 2 & 2 \end{array}\right) \tag{29}$$

$$(-1/4)L_1 + L_3 \to L_3 \implies \left(\begin{array}{ccc:c} 4 & 8 & 1 & 1 \\ 0 & 0 & 7/4 & 3/4 \\ 0 & -1 & 7/4 & 7/4 \end{array}\right) \tag{30}$$

$$L_2 \leftrightarrow L_3 \implies \left(\begin{array}{ccc:c} 4 & 8 & 1 & 1 \\ 0 & -1 & 7/4 & 7/4 \\ 0 & 0 & 7/4 & 3/4 \end{array}\right) \tag{31}$$

c. solve for x_3, x_2, x_1 in that order

$$(7/4)x_3 = 3/4 \implies x_3 = 3/7$$

$$-x_2 = (7/4)(1 - x_3) = 7/4 - 3/4 \implies x_2 = -1$$

$$4x_1 = 1 - x_3 - 8x_2 = 1 - 3/7 + 8 \implies x_1 = (1/4)(-3/7) + 1/4 + 2$$

d. How many pivot points, hence, pivot variables? 3
e. How many free variables are there? 0
f. How many solutions are there if any? one solution

(15pts) h. Put into canonical form. Note, pivot points should be the only non-zero number in its column.

$$-L_3 + L_2 \to L_2 \implies \left(\begin{array}{ccc|c} 4 & 8 & 1 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 7/4 & 3/4 \end{array}\right) \tag{32}$$

$$\begin{array}{c} (1/4)L_1 \to L_4 \\ (-1)L_2 \to L_2 \\ (4/7)L_3 \to L_3 \end{array} \implies \left(\begin{array}{ccc|c} 1 & 2 & 1/4 & 1/4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3/7 \end{array}\right) \tag{33}$$

$$(-1/4)L_3 + L_1 \to L_1 \implies \left(\begin{array}{ccc|c} 1 & 2 & 0 & (-1/4)(3/7) + 1/4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3/7 \end{array}\right) \tag{34}$$

$$(-2)L_2 + L_1 \to L_1 \implies \left(\begin{array}{ccc|c} 1 & 0 & 0 & (-1/4)(3/7) + 1/4 + 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3/7 \end{array}\right) \tag{35}$$

i. Put final solution in decimal form and compare with solution given by Matlab by applying the command **rref** to the original augmented matrix.

$$\mathrm{rref}\left([4, 8, 1, 1; 1, 2, 2, 1; 1, 1, 2, 2]\right) = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2.1429 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & .4286 \end{array}\right) \tag{36}$$

j. In Matlab, take an augmented matrix from one of the intermediate steps and apply rref. Is the solution different? No.

Problem 1. iii. Consider the following system of linear equations

$$x_1 + 2x_2 + x_3 = 1 \tag{37}$$
$$x_1 + 2x_3 = 2 \tag{38}$$
$$2x_1 + 2x_2 + 3x_3 = 3 \tag{39}$$

a. Put into augmented matrix form

$$\left(\begin{array}{ccc:c} 1 & 2 & 1 & 1 \\ 1 & 0 & 2 & 2 \\ 2 & 2 & 3 & 3 \end{array}\right) \tag{40}$$

b. Put in echelon form

$$(-1)L_1 + L_2 \to L_2 \implies \left(\begin{array}{ccc:c} 1 & 2 & 1 & 1 \\ 0 & -2 & 1 & 1 \\ 2 & 2 & 3 & 3 \end{array}\right) \tag{41}$$

$$(-2)L_1 + L_3 \to L_3 \implies \left(\begin{array}{ccc:c} 1 & 2 & 1 & 1 \\ 0 & -2 & 1 & 1 \\ 0 & -2 & 1 & 1 \end{array}\right) \tag{42}$$

$$(-1)L_2 + L_3 \to L_3 \implies \left(\begin{array}{ccc:c} 1 & 2 & 1 & 1 \\ 0 & -2 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \tag{43}$$

c. solve for x_3, x_2, x_1 in that order

$$x_3 \text{ is a free variable and can be anything}$$

$$-2x_2 = 1 - x_3 \implies x_2 = x_3/2 - 1/2$$
$$x_1 = 1 - x_3 - 2x_2 = 1 - x_3 - x_3 + 1 \implies x_1 = 2 - 2x_3$$

d. How many pivot points, hence, pivot variables? 2
e. How many free variables are there? 1
f. How many solutions are there if any? infinite solutions
h. Put into canonical form. Note, pivot points should be the only non-zero number in its column.

$$(-1/2)L_2 \to L_2 \implies \left(\begin{array}{ccc:c} 1 & 2 & 1 & 1 \\ 0 & 1 & -1/2 & -1/2 \end{array}\right) \tag{44}$$

$$(-2)L_2 + L_1 \to L_1 \implies \left(\begin{array}{ccc:c} 1 & 0 & 2 & 2 \\ 0 & 1 & -1/2 & -1/2 \end{array}\right) \tag{45}$$

i. Put final solution in decimal form and compare with solution given by Matlab by applying the command **rref** to the original augmented matrix.

$$\text{rref}\,([1,2,1,1;1,0,2,2;2,2,3,3]) = \left(\begin{array}{ccc:c} 1 & 0 & 2 & 2 \\ 0 & 1 & -.5 & -.5 \end{array}\right) \tag{46}$$

j. In Matlab, take an augmented matrix from one of the intermediate steps and apply rref. Is the solution different? No.

Problem 1. iv. Consider the following system of linear equations

$$2x_1 + x_2 = 10 \tag{47}$$
$$x_3 + x_2 = 2 \tag{48}$$
$$2x_1 + 2x_2 + x_3 = 12 \tag{49}$$

(10pts) a. Put into augmented matrix form

$$\left(\begin{array}{ccc|c} 2 & 1 & 0 & 10 \\ 0 & 1 & 1 & 2 \\ 2 & 2 & 1 & 12 \end{array}\right) \tag{50}$$

(10pts) b. Put in echelon form

$$(-1)L_1 + L_3 \to L_3 \implies \left(\begin{array}{ccc|c} 2 & 1 & 0 & 10 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & -2 \end{array}\right) \tag{51}$$

$$(-1)L_2 + L_3 \to L_3 \implies \left(\begin{array}{ccc|c} 2 & 1 & 0 & 10 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right) \tag{52}$$

c. solve for x_3, x_2, x_1 in that order

$$x_3 \text{ is a free variable and can be anything}$$

$$x_2 = 2 - x_3$$
$$2x_1 = 10 - x_2 = 10 - 2 + x_3 \implies x_1 = 4 + x_3/2$$

d. How many pivot points, hence, pivot variables? 2
e. How many free variables are there? 1
f. How many solutions are there if any? infinite solutions
(15pts) h. Put into canonical form. Note, pivot points should be the only non-zero number in its column.

$$(-1)L_2 + L_1 \to L_1 \implies \left(\begin{array}{ccc|c} 2 & 0 & -1 & 8 \\ 0 & 1 & 1 & 2 \end{array}\right) \tag{53}$$

$$(1/2)L_1 \to L_1 \implies \left(\begin{array}{ccc|c} 1 & 0 & -1/2 & 4 \\ 0 & 1 & 1 & 2 \end{array}\right) \tag{54}$$

i. Put final solution in decimal form and compare with solution given by Matlab by applying the command **rref** to the original augmented matrix.

$$\text{rref}\left([2,1,0,10;0,1,1,2;2,2,1,12]\right) = \left(\begin{array}{ccc|c} 1 & 0 & -.5 & 4 \\ 0 & 1 & 1 & 2 \end{array}\right) \tag{55}$$

j. In Matlab, take an augmented matrix from one of the intermediate steps and apply rref. Is the solution different? No.

Problem 2 **(40 pts - 5pts/each)**

For the following matrices, determine if they are in echelon form. If the matrix is not in echelon form, perform row operations until it is in echelon form. Determine if there is one solution, no solution or infinitely many solutions. If there are free variables, which are the free variables? Which variables are the pivot variables?

$$\text{a.}\begin{pmatrix} \begin{array}{cccc|c} 4 & 8 & 1 & 1 & 2 \\ 0 & 2 & 2 & 1 & 1 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 2 & 2 \end{array} \end{pmatrix} \text{b.}\begin{pmatrix} \begin{array}{cccc|c} 0 & 0 & 2 & 2 & 2 \\ 4 & 8 & 1 & 1 & 2 \\ 0 & 2 & 2 & 1 & 1 \\ 0 & 0 & 0 & 2 & 2 \end{array} \end{pmatrix} \text{c.}\begin{pmatrix} \begin{array}{cccc|c} 0 & 0 & 2 & 2 & 2 \\ 4 & 8 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 & 2 \end{array} \end{pmatrix} \text{d.}\begin{pmatrix} \begin{array}{cccc|c} 4 & 8 & 1 & 1 & 2 \\ 0 & 1 & 2 & 2 & 2 \\ 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 2 \end{array} \end{pmatrix} \tag{56}$$

$$\text{e.}\begin{pmatrix} \begin{array}{cccc|c} 0 & 8 & 1 & 1 & 2 \\ 0 & 2 & 2 & 1 & 1 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 2 \end{array} \end{pmatrix} \text{f.}\begin{pmatrix} \begin{array}{cccc|c} 4 & 8 & 1 & 1 & 2 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 2 & 2 & 2 \end{array} \end{pmatrix} \text{g.}\begin{pmatrix} \begin{array}{cccc|c} 0 & 0 & 0 & 1 & 1 \\ 4 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 10 & 10 \\ 0 & 0 & 0 & 2 & 2 \end{array} \end{pmatrix} \text{h.}\begin{pmatrix} \begin{array}{cccc|c} 0 & 0 & 0 & 2 & 2 \\ 4 & 0 & 1 & 1 & 2 \\ 0 & 8 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 & 10 \end{array} \end{pmatrix} \tag{57}$$

a. The system is in echelon form and so a solution exists. It is triangular form and so it has a unique solution. There are 4 pivot variables and no free variables.

b. Echelon form is exactly part a. with the same results.

c. Switch the first two rows to put in echelon form. This implies a solution exists (no inconsistent equations). There are 3 pivot variables x_1, x_3, x_4 and 1 free variable x_2. There are infinite solutions.

d. Apply $-2L_2 + L_3 \to L_3$ to achieve echelon form

$$\begin{pmatrix} \begin{array}{cccc|c} 4 & 8 & 1 & 1 & 2 \\ 0 & 1 & 2 & 2 & 2 \\ 0 & 0 & -4 & -4 & -3 \\ 0 & 0 & 0 & 2 & 2 \end{array} \end{pmatrix}$$

There are 4 pivot variables and one solution.

e. The system has an inconsistent equation and so it has no solution.

f. Apply $-L_2 + L_3 \to L_3$ to achieve echelon form

$$\begin{pmatrix} \begin{array}{cccc|c} 4 & 8 & 1 & 1 & 2 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array} \end{pmatrix}$$

There are 3 pivot variables x_1, x_3, x_4, one free variable x_2 and infinite solutions.

g. Echelon form

$$\begin{pmatrix} \begin{array}{cccc|c} 4 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{array} \end{pmatrix}$$

There are 2 pivot variables x_1, x_4, two free variables x_2, x_3 and infinite solutions.

h. The system is inconsistent and has no solution.