AMS10 HW3-Grading Rubric

*On all problems you can use Matlab to verify your answer but you must show work unless otherwise indicated.

Problem 1 (20pts- 10pts/each) Consider the following set of vectors

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \right\} = \{v_1, v_2\}.$$

$Span(\{v_1, v_2\})$ is the set of all linear combinations of $\{v_1, v_2\}$:

$$k_1 v_1 + k_2 v_2 : k_1, k_2 \in \mathbb{R}.$$

Let $\vec{b} = [a, b, c]^T \in Span(S)$. Note, we can rewrite this as

$$[v_1, v_2] \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

a. Put the augmented matrix $[A|\vec{b}]$ in row canonical form, carrying through variables a, b, c.

$$\left[\begin{array}{cc:c} 1 & -1 & a \\ 0 & 2 & b \\ 0 & 0 & c \end{array}\right] \rightarrow \left[\begin{array}{cc:c} 1 & 0 & b/2 + a \\ 0 & 1 & b/2 \\ 0 & 0 & c \end{array}\right]$$

b. Define the set of vectors that is not in the span of $\{v_1, v_2\}$ If $c \neq 0$, then the system is inconsistent and has no solution.

$$\left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : c \neq 0 \right\}$$

Problem 2 (25pts -5pts/each) Consider the following system

$$x_1 + x_2 + 2x_3 = 1$$
$$x_1 + x_3 = 0$$

a. Find the row canonical form of the corresponding augmented matrix. How many solutions, if any, does this system have?

$$\text{rref}\left(\left[\begin{array}{ccc:c} 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 0 \end{array}\right] \right) = \left[\begin{array}{ccc:c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right]$$

The system has infinite solutions.

b. Write down the reduced system (obtained after reduction to row canonical form) as a linear combination of vectors:

$$\vec{a}_1 x_1 + \vec{a}_2 x_2 + \vec{a}_3 x_3 = \vec{b}.$$

What is $\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{b}$?

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 1 \end{bmatrix} x_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

c. Does there exist scalars k_1, k_2, k_3 such that

$$k_1\vec{a}_1 + k_2\vec{a}_2 + k_3\vec{a}_3 = 0?$$

If so, give an example. Is the solution unique?

One possible solution is $k_1 = -1, k_2 = -1, k_3 = 1$. This solution is not unique because the set of vectors are linearly dependent.

d. Are the vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3$ linearly dependent or independent? Why?

Credit for stating that the vectors are linearly dependent. The first two vectors span the entire $\mathbb{R}^2$ domain and so three vectors are linearly dependent. The third vector can be expressed as a linear combination of the first two.

e. Does there exist a different vector $\vec{b}$ for which no solution exists? Why?

No because the first two vectors span $\mathbb{R}^2$.

Problem 3 (5pts) Consider the following vectors

$$v_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ -2 \\ -4 \end{bmatrix}, v_3 = \begin{bmatrix} -2 \\ 6 \\ 14 \end{bmatrix}.$$

a. What is the rank of matrix $A = [v_1, v_2, v_3]$? What does this say about the dimension of $colsp(A) = span(v_1, v_2, v_3)$?

$$rref(A) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

The rank(A)= 2, since the echelon form has two pivot points. This means there are maximum two linearly independent column vectors and so the dim(colsp(A))=2, which is also the dim(Im(A)) and equivalently the dim(span($\{v_1, v_2, v_3\}$)).

Problem 4 (30pts-5pts/each) Let matrix A be composed of

$$v_1 = \begin{bmatrix} 9 \\ -4 \\ 5 \\ 5 \end{bmatrix}, v_2 = \begin{bmatrix} -19 \\ 2 \\ -19 \\ -2 \end{bmatrix}, v_3 = \begin{bmatrix} -34 \\ 28 \\ -2 \\ -36 \end{bmatrix}, v_4 = \begin{bmatrix} 1 \\ 6 \\ 9 \\ -8 \end{bmatrix}.$$

In this problem you may use Matlab to reduce matrices to row canonical form.

a. Reduce the matrix $A = [v_1, v_2, v_3, v_4]$ to row canonical form. Are the vectors v_1, v_2, v_3, v_4 linearly dependent or independent?

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & -8 & -2 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The vectors are linearly dependent. The rank of the matrix is two but there are four vectors. Rank is equal to the maximum number of linearly independent vectors.

b. Reduce the matrix $A = [v_1, v_2]$ to row canonical form. Are the vectors v_1, v_2 linearly dependent or independent? How does the row canonical form compare to the first two columns for the reduced matrix in part a.

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The vectors are linearly independent since the matrix has rank 2 and the number of column vectors are two. The row canonical form is identical to the first two columns.

c. Reduce the matrix $A = [v_1, v_2, v_3]$ to row canonical form. Are the vectors v_1, v_2, v_3 linearly dependent or independent? How does the row canonical form compare to the first three columns for the reduced matrix in part a.

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The vectors are linearly dependent since the matrix is rank 2 and there are three vectors. The row canonical form is identical to the first three columns.

d. Reduce the matrix $A = [v_3, v_4]$ to row canonical form. Are the vectors v_3, v_4 linearly dependent or independent?

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The vectors are linearly independent since the matrix has rank 2 and the number of column vectors are two.

e. Give a basis for the span$\{v_1, v_2, v_3, v_4\}$.

A basis can be any set of two linearly independent vectors from the set. A basis can be $\{v_1, v_2\}$ or $\{v_3, v_4\}$.

f. What is the $dim(cols(A))$? Will any subset of three vectors always be linearly dependent?

Any subset of three vectors will be linearly dependent. The dimension of the span of the vectors is two.

Problem 5 (15pts-5pts each) Consider the following set of vectors

$$v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -2 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ -3 \\ -2 \end{bmatrix}, v_4 = \begin{bmatrix} -2 \\ -2 \\ 2 \\ -3 \\ 8 \end{bmatrix}.$$

a.Find the dimension of the subspace spanned by the vectors

One way to determine the dimension of the subspace is to find the rank of a matrix composed of the vectors. We construct a matrix composed of the column vectors and find the echelon form

$$\text{rref}([v_1, v_2, v_3, v_4]) = \begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The matrix has rank(A)= 3 and so the dim(span($\{v_1, v_2, v_3, v_4\}$))=3 .

b. Find a basis for the subspace spanned by the vectors

We can construct a matrix where the row vectors are the given vectors and reduce to row canonical form:

$$\text{rref}(\left[v_1^T; v_2^T; v_3^T; v_4^T\right]) = \begin{bmatrix} 1 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & -1.5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

then a basis would consist of the nonzero rows

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1.5 \\ 0 \end{bmatrix} \right\}.$$

Equivalently one can take the vectors corresponding to the pivot points of the reduced matrix composed of the column vectors

$$\text{rref}([v_1, v_2, v_3, v_4]) = \begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which tells us $\{v_1, v_2, v_3\}$ is a basis.

c.What is the rank of the matrix $A = [v_1, v_2, v_3, v_4]$?

We found in part a. that rank(A)=3.

***5pts for submission**