AMS10 HW4 Grading Rubric

*On all problems you can use Matlab to verify your answer but you must show work.

Problem 1 (35pts 5pts/each faithful effort credit) Let

$$A = \begin{bmatrix} 1 & -4 \\ 2 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 2 \\ 0 & 7 \end{bmatrix}$$

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \; \vec{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \; c = 2$$

a. Show that $A\vec{v}_1 + A\vec{v}_2 = A(\vec{v}_1 + \vec{v}_2)$

$$A\vec{v}_1 + A\vec{v}_2 = \begin{bmatrix} 1 & -4 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 1 & -4 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -5 \\ 6 \end{bmatrix}$$

$$A(\vec{v}_1 + \vec{v}_2) = \begin{bmatrix} 1 & -4 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -5 \\ 6 \end{bmatrix}$$

b. Show that $AB \neq BA$

$$AB = \begin{bmatrix} 1 & -4 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 6 & -13 \\ 4 & -2 \end{bmatrix}$$

$$BA = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & -4 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -8 \\ 5 & 4 \end{bmatrix}$$

c. Show that $A\vec{v}_1 + B\vec{v}_1 = (A+B)\vec{v}_1$

$$A\vec{v}_1 + B\vec{v}_1 = \begin{bmatrix} 1 & -4 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -14 \\ 14 \end{bmatrix}$$

$$(A+B)\vec{v}_1 = \begin{bmatrix} 3 & -5 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -14 \\ 14 \end{bmatrix}$$

d. Show that $A(c\vec{v}_1) = c(A\vec{v}_1) = (cA)\vec{v}_1$. Note that $[cA]_{ij} = c[A]_{ij}$.

$$A(c\vec{v}_1) = \begin{bmatrix} 1 & -4 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 4 \\ 8 \end{bmatrix} = \begin{bmatrix} -28 \\ 8 \end{bmatrix}$$

$$c(A\vec{v}_1) = 2\begin{bmatrix} -14 \\ 4 \end{bmatrix} = \begin{bmatrix} -28 \\ 8 \end{bmatrix}$$

$$(cA)\vec{v}_1 = \begin{bmatrix} 2 & -8 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -28 \\ 8 \end{bmatrix}$$

e. Show that $A(B+C) = AB + AC$.

$$A(B+C) = \begin{bmatrix} 1 & -4 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 10 \end{bmatrix} = \begin{bmatrix} 7 & -39 \\ 6 & 2 \end{bmatrix}$$

$$AB + AC = \begin{bmatrix} 6 & -13 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} 1 & -26 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 7 & -39 \\ 6 & 2 \end{bmatrix}$$

f. Show that $(A+B)C = AC + BC$.

$$(A+B)C = \begin{bmatrix} 3 & -5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 3 & -29 \\ 1 & 23 \end{bmatrix}$$

$$AC + BC = \begin{bmatrix} 1 & -26 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -1 & 19 \end{bmatrix} = \begin{bmatrix} 3 & -29 \\ 1 & 23 \end{bmatrix}$$

g. Show that $c(AB) = (cA)B = (AB)c = A(Bc)$.

$$c(AB) = 2\begin{bmatrix} 6 & -13 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} 12 & -26 \\ 8 & -4 \end{bmatrix}$$

$$(cA)B = \begin{bmatrix} 2 & -8 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 12 & -26 \\ 8 & -4 \end{bmatrix}$$

$$(AB)c = \begin{bmatrix} 6 & -13 \\ 4 & -2 \end{bmatrix} 2 = \begin{bmatrix} 12 & -26 \\ 8 & -4 \end{bmatrix}$$

$$A(Bc) = \begin{bmatrix} 1 & -4 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ -2 & 6 \end{bmatrix} = \begin{bmatrix} 12 & -26 \\ 8 & -4 \end{bmatrix}$$

Problem 2 (15pts 5pts/each faithful effort credit) For the following pairs of A and $\vec{v}$, first determine whether the product $A\vec{v}$ is defined. If it is, what is the expected dimension of the product? Calculate the product and verify the dimension.

a.

$$A = [1\ 2\ 3\ 4]\,,\ \vec{v} = \begin{bmatrix} 1 \\ 8 \\ 0 \\ 2 \end{bmatrix}$$

A is (1×4) and $\vec{v}$ is (4×1), so $A\vec{v}$ is (1×1)

$$A\vec{v} = [1\ 2\ 3\ 4] \begin{bmatrix} 1 \\ 8 \\ 0 \\ 2 \end{bmatrix} = 25$$

b.

$$A = \begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & 8 & -3 \end{bmatrix},\ \vec{v} = \begin{bmatrix} 1 \\ 8 \\ 0 \\ 2 \end{bmatrix}$$

A is (2×4) and $\vec{v}$ is (4×1), so $A\vec{v}$ is (2×1)

$$A\vec{v} = \begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & 8 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 8 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -7 \\ 2 \end{bmatrix}$$

c.

$$A = \begin{bmatrix} 1 & 4 \\ 0 & -3 \\ 2 & 1 \end{bmatrix}, \ \vec{v} = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}$$

A is (3×2) and $\vec{v}$ is (3×1), so $A\vec{v}$ is not defined.

Problem 3 (15pts 5pts/each faithful effort credit) For the following pairs of A and B, first determine whether the product AB is defined. If it is, what is the expected dimension of the product? Calculate the product and verify the dimension.

a.

$$A = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 3 & 4 & -1 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 1 & 0 & 7 \\ 2 & -4 & 9 \\ -1 & 3 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

A is (2×4) and B is (4×3), so AB is (2×3)

$$AB = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 3 & 4 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 7 \\ 2 & -4 & 9 \\ -1 & 3 & 0 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} -3 & 9 & -9 \\ 12 & -19 & 57 \end{bmatrix}$$

b.

$$A = \begin{bmatrix} 1 & -4 & 3 \\ 0 & 2 & 1 \end{bmatrix}, \ B = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -4 & 1 & 5 \\ -2 & -3 & 1 \end{bmatrix}$$

A is (2×3) and B is (4×3), so AB is not defined.

c.

$$A = \begin{bmatrix} -1 & 0 & 2 & 1 \\ 0 & -1 & 5 & 2 \\ 3 & 7 & 0 & 3 \end{bmatrix}, \ B = \begin{bmatrix} 4 & 0 \\ 0 & -1 \\ 2 & 1 \\ 3 & 2 \end{bmatrix}$$

A is (3×4) and B is (4×2), so AB is (3×2)

$$AB = \begin{bmatrix} -1 & 0 & 2 & 1 \\ 0 & -1 & 5 & 2 \\ 3 & 7 & 0 & 3 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & -1 \\ 2 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 16 & 10 \\ 21 & -1 \end{bmatrix}$$

Problem 4 (5pts) Consider the following system

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 1 \\ x_1 + x_3 &= 0 \end{aligned}$$

a. Find the row canonical form of the corresponding augmented matrix. How many solutions, if any, does this system have?

$$\text{rref}\left(\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 0 \end{array} \right] \right) = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{array} \right]$$

The system has infinite solutions.

b. Write down the reduced system (obtained after reduction to row canonical form) as a linear combination of vectors:

$$\vec{a}_1 x_1 + \vec{a}_2 x_2 + \vec{a}_3 x_3 = \vec{b}.$$

What is $\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{b}$?

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 1 \end{bmatrix} x_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

c. Does there exist scalars k_1, k_2, k_3 such that

$$k_1 \vec{a}_1 + k_2 \vec{a}_2 + k_3 \vec{a}_3 = 0?$$

If so, give an example. Is the solution unique?

One possible solution is $k_1 = 0, k_2 = 1, k_3 = 0$. This solution is not unique because the set of vectors are linearly dependent.

d. **(5pts)** Are the vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3$ linearly dependent or independent? Why?

Credit for stating that the vectors are linearly dependent. The first two vectors span the entire $\mathbb{R}^2$ domain and so three vectors are linearly dependent. The third vector can be expressed as a linear combination of the first two.

e. Does there exist a different vector $\vec{b}$ for which no solution exists? Why?

No because the first two vectors span $\mathbb{R}^2$.

f. Put in matrix multiplication form $A\vec{x} = \vec{b}$. Augment the system with

$$0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 0,$$

which doesn't change the system and put it in matrix multiplication form again. What happens if you type in inv(A) into Matlab in both cases? Why?

In the first case

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

and the inverse cannot be computed because the matrix is not a square matrix. In the second case

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

the inverse does not exist because the rows/columns are linearly dependent.

Problem 5 (10 pts) Consider the following system

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 1 \\ x_1 + x_3 &= 0 \\ 2x_1 + x_2 + 3x_3 &= 1 \end{aligned}$$

a. Is the system above linearly dependent or independent? Show using any method.

The system is linearly dependent because the third equation is a sum of the first two equations. We can also see this by looking at the reduced augmented matrix

$$\text{rref}\left(\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 3 & 1 \end{array}\right]\right) = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Since the third row reduced to all zeros, this tells us the original set of row vectors were linearly dependent.

b. Put the system in matrix multiplication form $A\vec{x} = \vec{b}$. Does the inverse A^{-1} exist? Why or why not?

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

The inverse does not exist because the rows/columns are linearly dependent

c. (10pts) Use Matlab to find the row canonical form and write the output. How does the row canonical form compare to that of the system above? Explain why it's the same or different.

5pts for the right row canonical form

$$\text{rref}\left(\begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 3 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

5pts if it agrees with Problem 4a. The row canonical form is identical to that in Problem 4. This is because adding a third equation that was a linear combination of the first two didn't change the system.

Problem 6 (20 pts) Consider the following system

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 1 \\ x_1 + x_3 &= 0 \\ 2x_1 + x_2 + 3x_3 &= 1 \\ x_1 - 5x_2 &= 0 \end{aligned}$$

a. There are more equations than unknowns. More equations than unknowns always implies a linearly dependent set if not inconsistent. Apply Gaussian elimination steps (towards row echelon form) until the system reduces to 3 equations. Note that the remaining equations in echelon form are always linearly independent. No need to fully reduce system.

The third equation is the sum of the first two, so I remove it from the system of equations. This is a short cut path.

b. (5pts) Put the partially reduced system (3 equations) in matrix multiplication form $A\vec{x} = \vec{b}$. What is A, $\vec{x}$, and $\vec{b}$?

Credit given for a 3×3 A matrix.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & -5 & 0 \end{bmatrix}$$

with corresponding $\vec{b}$ vector

$$\vec{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Answers can vary.

c. (5 pts) Calculate the inverse A^{-1} by reducing the left side of the augmented matrix $[A|I]$ to row canonical form, where I is the identity matrix. Solution can be verified using the function inv in Matlab.

Credit given if the augmented matrix $[A|I]$ is transformed into $[I|A^{-1}]$ using elementary row operations The inverse of A is

$$A^{-1} = \begin{bmatrix} -1.25 & 2.5 & -.25 \\ -.25 & .5 & -.25 \\ 1.25 & -1.5 & .25 \end{bmatrix}$$

Answers can vary.

d. (10 pts) Find the solution $\vec{x}$ using the calculated inverse. Show work.

$$\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} -1.25 & 2.5 & -.25 \\ -.25 & .5 & -.25 \\ 1.25 & -1.5 & .25 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1.25 \\ -.25 \\ 1.25 \end{bmatrix}.$$

e. Verify your answer by applying the Matlab function rref on the augmented matrix of the original system to show that the answers are the same. Write down the output.

$$\text{rref}\left(\begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 3 & 1 \\ 1 & -5 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 & -1.25 \\ 0 & 1 & 0 & -.25 \\ 0 & 0 & 1 & 1.25 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$