AMS10 HW4

Problem 1 (not graded but please make note of the properties) Let

$$A = \begin{bmatrix} 1 & -4 \\ 2 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 2 \\ 0 & 7 \end{bmatrix}$$

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \; \vec{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \; c = 2$$

a. Show that $A\vec{v}_1 + A\vec{v}_2 = A(\vec{v}_1 + \vec{v}_2)$
b. Show that $AB \neq BA$
c. Show that $A\vec{v}_1 + B\vec{v}_1 = (A + B)\vec{v}_1$
d. Show that $A(c\vec{v}_1) = c(A\vec{v}_1) = (cA)\vec{v}_1$. Note that $[cA]_{ij} = c[A]_{ij}$.
e. Show that $A(B + C) = AB + AC$.
f. Show that $(A + B)C = AC + BC$.
g. Show that $c(AB) = (cA)B = (AB)c = A(Bc)$.

Problem 2 For the following pairs of A and $\vec{v}$, first determine whether the product $A\vec{v}$ is defined. If it is, what is the expected dimension of the product? Calculate the product and verify the dimension.

a.

$$A = [1\ 2\ 3\ 4], \; \vec{v} = \begin{bmatrix} 1 \\ 8 \\ 0 \\ 2 \end{bmatrix}$$

b.

$$A = \begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & 8 & -3 \end{bmatrix}, \; \vec{v} = \begin{bmatrix} 1 \\ 8 \\ 0 \\ 2 \end{bmatrix}$$

c.

$$A = \begin{bmatrix} 1 & 4 \\ 0 & -3 \\ 2 & 1 \end{bmatrix}, \; \vec{v} = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}$$

Problem 3 For the following pairs of A and B, first determine whether the product AB is defined. If it is, what is the expected dimension of the product? Calculate the product and verify the dimension.

a.

$$A = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 3 & 4 & -1 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 0 & 7 \\ 2 & -4 & 9 \\ -1 & 3 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

b.

$$A = \begin{bmatrix} 1 & -4 & 3 \\ 0 & 2 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -4 & 1 & 5 \\ -2 & -3 & 1 \end{bmatrix}$$

c.

$$A = \begin{bmatrix} -1 & 0 & 2 & 1 \\ 0 & -1 & 5 & 2 \\ 3 & 7 & 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 0 \\ 0 & -1 \\ 2 & 1 \\ 3 & 2 \end{bmatrix}$$

Problem 4 Consider the following vectors

$$v_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ -2 \\ -4 \end{bmatrix}, v_3 = \begin{bmatrix} -2 \\ 6 \\ 14 \end{bmatrix}.$$

a. Note that solving $A\vec{x} = \vec{0}$, gives the linear combination of the vectors v_1, v_2, v_3 that maps to the zero vector. Hence, it gives vectors $\vec{x}$ in the nullspace of A. Use the reduced matrix of A (you may use rref([A|0]) in Matlab) to find a vector $\vec{x}$ that satisfies the system of equations and verify it is in the null space of A by showing $A\vec{x} = 0$.
b. Use the Rank-Nulliity theorem to determine the dimension of Ker(A). Does this agree with part **a**?

Problem 5 Given the matrices

$$A = \begin{bmatrix} 5 & 4 \\ -3 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 1 \\ -2 & 1 & 4 \\ 2 & 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 0 & -3 & 1 \\ 1 & 1 & -2 \\ 0 & 5 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 0 & -3 & 1 & 0 \\ 1 & 0 & -1 & 2 \\ 0 & 0 & 1 & 5 \\ 3 & -1 & 0 & 2 \end{bmatrix}$$

calculate the following determinants:
a. $\det(A)$
b. $\det(B)$
c. $\det(C)$
d. $\det(BC)$ (Hint: Can use results from b. and c.)
e. $\det(CB)$ (Hint: Can use results from b. and c.)
f. (optional) $\det(D)$
You can verify answers using Matlab but show your work.

Problem 6
a. Find the determinant of the following matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 2 \\ 2 & 0 & 1 \end{bmatrix}$$

b. Find a basis for the Image of A. Recall, the image of A is the span of the column vectors. What is the dim(Im(A))?
c. Find a basis for the Kernel of A. What is dim(Ker(A))?
d. Does A^{-1} exist?

Problem 7 Consider the following system

$$
\begin{aligned}
x_1 + x_2 + 2x_3 &= 1 \\
x_1 + x_3 &= 0 \\
2x_1 + x_2 + 3x_3 &= 1 \\
x_1 - 5x_2 &= 0
\end{aligned}
$$

a. There are more equations than unknowns. More equations than unknowns always implies a linearly dependent set if not inconsistent. Apply Gaussian elimination steps (towards row echelon form) until the system reduces to 3 equations. Note that the remaining equations in echelon form are always linearly independent. No need to fully reduce system.

b. Put the partially reduced system (3 equations) in matrix multiplication form $A\vec{x} = \vec{b}$. What is A, $\vec{x}$, and $\vec{b}$?

c. Calculate A^{-1} by reducing the left side of the augmented matrix $[A|I]$ to row canonical form, where I is the identity matrix. Solution can be verified using the function inv in Matlab. What is the dim(Ker(A))?

d. Find the solution $\vec{x}$ using the calculated inverse. Show work.

e. (not graded) Verify your answer by applying the Matlab function rref on the augmented matrix of the original system to show that the answers are the same. Write down the output.

Extra optional practice problems: Compute the inverse for matrices in **a-f** of **Problem 5**. You can verify your answer with the Matlab inverse command function.

AMS10 HW4 Solutions

Problem 1 (not graded) Let

$$A = \begin{bmatrix} 1 & -4 \\ 2 & 0 \end{bmatrix},\ B = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix},\ C = \begin{bmatrix} 1 & 2 \\ 0 & 7 \end{bmatrix}$$

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 4 \end{bmatrix},\ \vec{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix},\ c = 2$$

a. Show that $A\vec{v}_1 + A\vec{v}_2 = A(\vec{v}_1 + \vec{v}_2)$

$$A\vec{v}_1 + A\vec{v}_2 = \begin{bmatrix} 1 & -4 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 1 & -4 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -5 \\ 6 \end{bmatrix}$$

$$A(\vec{v}_1 + \vec{v}_2) = \begin{bmatrix} 1 & -4 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -5 \\ 6 \end{bmatrix}$$

b. Show that $AB \neq BA$

$$AB = \begin{bmatrix} 1 & -4 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 6 & -13 \\ 4 & -2 \end{bmatrix}$$

$$BA = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & -4 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -8 \\ 5 & 4 \end{bmatrix}$$

c. Show that $A\vec{v}_1 + B\vec{v}_1 = (A + B)\vec{v}_1$

$$A\vec{v}_1 + B\vec{v}_1 = \begin{bmatrix} 1 & -4 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -14 \\ 14 \end{bmatrix}$$

$$(A + B)\vec{v}_1 = \begin{bmatrix} 3 & -5 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -14 \\ 14 \end{bmatrix}$$

d. Show that $A(c\vec{v}_1) = c(A\vec{v}_1) = (cA)\vec{v}_1$. Note that $[cA]_{ij} = c[A]_{ij}$.

$$A(c\vec{v}_1) = \begin{bmatrix} 1 & -4 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 4 \\ 8 \end{bmatrix} = \begin{bmatrix} -28 \\ 8 \end{bmatrix}$$

$$c(A\vec{v}_1) = 2\begin{bmatrix} -14 \\ 4 \end{bmatrix} = \begin{bmatrix} -28 \\ 8 \end{bmatrix}$$

$$(cA)\vec{v}_1 = \begin{bmatrix} 2 & -8 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -28 \\ 8 \end{bmatrix}$$

e. Show that $A(B + C) = AB + AC$.

$$A(B + C) = \begin{bmatrix} 1 & -4 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 10 \end{bmatrix} = \begin{bmatrix} 7 & -39 \\ 6 & 2 \end{bmatrix}$$

$$AB + AC = \begin{bmatrix} 6 & -13 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} 1 & -26 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 7 & -39 \\ 6 & 2 \end{bmatrix}$$

f. Show that $(A + B)C = AC + BC$.

$$(A+B)C = \begin{bmatrix} 3 & -5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 3 & -29 \\ 1 & 23 \end{bmatrix}$$

$$AC + BC = \begin{bmatrix} 1 & -26 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -1 & 19 \end{bmatrix} = \begin{bmatrix} 3 & -29 \\ 1 & 23 \end{bmatrix}$$

g. Show that $c(AB) = (cA)B = (AB)c = A(Bc)$.

$$c(AB) = 2 \begin{bmatrix} 6 & -13 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} 12 & -26 \\ 8 & -4 \end{bmatrix}$$

$$(cA)B = \begin{bmatrix} 2 & -8 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 12 & -26 \\ 8 & -4 \end{bmatrix}$$

$$(AB)c = \begin{bmatrix} 6 & -13 \\ 4 & -2 \end{bmatrix} 2 = \begin{bmatrix} 12 & -26 \\ 8 & -4 \end{bmatrix}$$

$$A(Bc) = \begin{bmatrix} 1 & -4 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ -2 & 6 \end{bmatrix} = \begin{bmatrix} 12 & -26 \\ 8 & -4 \end{bmatrix}$$

Problem 2 (15pts- 5pts each) For the following pairs of A and $\vec{v}$, first determine whether the product $A\vec{v}$ is defined. If it is, what is the expected dimension of the product? Calculate the product and verify the dimension.

a.

$$A = [1\ 2\ 3\ 4]\,,\ \vec{v} = \begin{bmatrix} 1 \\ 8 \\ 0 \\ 2 \end{bmatrix}$$

A is (1×4) and $\vec{v}$ is (4×1), so $A\vec{v}$ is (1×1)

$$A\vec{v} = [1\ 2\ 3\ 4] \begin{bmatrix} 1 \\ 8 \\ 0 \\ 2 \end{bmatrix} = 25$$

b.

$$A = \begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & 8 & -3 \end{bmatrix},\ \vec{v} = \begin{bmatrix} 1 \\ 8 \\ 0 \\ 2 \end{bmatrix}$$

A is (2×4) and $\vec{v}$ is (4×1), so $A\vec{v}$ is (2×1)

$$A\vec{v} = \begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & 8 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 8 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -7 \\ 2 \end{bmatrix}$$

c.

$$A = \begin{bmatrix} 1 & 4 \\ 0 & -3 \\ 2 & 1 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}$$

A is (3×2) and $\vec{v}$ is (3×1), so $A\vec{v}$ is not defined.

Problem 3 (15pts- 5pts each) For the following pairs of A and B, first determine whether the product AB is defined. If it is, what is the expected dimension of the product? Calculate the product and verify the dimension.

a.

$$A = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 3 & 4 & -1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 7 \\ 2 & -4 & 9 \\ -1 & 3 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

A is (2×4) and B is (4×3), so AB is (2×3)

$$AB = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 3 & 4 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 7 \\ 2 & -4 & 9 \\ -1 & 3 & 0 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} -3 & 9 & -9 \\ 12 & -19 & 57 \end{bmatrix}$$

b.

$$A = \begin{bmatrix} 1 & -4 & 3 \\ 0 & 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -4 & 1 & 5 \\ -2 & -3 & 1 \end{bmatrix}$$

A is (2×3) and B is (4×3), so AB is not defined.

c.

$$A = \begin{bmatrix} -1 & 0 & 2 & 1 \\ 0 & -1 & 5 & 2 \\ 3 & 7 & 0 & 3 \end{bmatrix}, B = \begin{bmatrix} 4 & 0 \\ 0 & -1 \\ 2 & 1 \\ 3 & 2 \end{bmatrix}$$

A is (3×4) and B is (4×2), so AB is (3×2)

$$AB = \begin{bmatrix} -1 & 0 & 2 & 1 \\ 0 & -1 & 5 & 2 \\ 3 & 7 & 0 & 3 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & -1 \\ 2 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 16 & 10 \\ 21 & -1 \end{bmatrix}$$

Problem 4 (10pts- 5pts each) Consider the following vectors

$$v_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ -2 \\ -4 \end{bmatrix}, v_3 = \begin{bmatrix} -2 \\ 6 \\ 14 \end{bmatrix}.$$

a. Note that solving $A\vec{x} = \vec{0}$, gives the linear combination of the vectors v_1, v_2, v_3 that maps to the zero vector. Hence, it gives vectors $\vec{x}$ in the nullspace of A. Use the reduced matrix of A (you may use rref([A|0]) in Matlab) to find a vector $\vec{x}$ that satisfies the system of equations and verify it is in the null space of A by showing $A\vec{x} = 0$. The reduced augmented matrix is

$$rref([A,0]) = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

A solution is given by

$$x_3 : \text{free variable} \quad (1)$$
$$x_2 = 3x_3 \quad (2)$$
$$x_1 = -2x_3 \quad (3)$$

and, hence, all vectors of the form

$$\vec{x} = \begin{bmatrix} -2x_3 \\ 3x_3 \\ x_3 \end{bmatrix}.$$

Choosing $\vec{x} = [-2, 3, 1]^T$, calculate $A\vec{x}$

$$\begin{bmatrix} -1 & 0 & -2 \\ 0 & -2 & 6 \\ 1 & -4 & 14 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

b. Use the Rank-Nulliity theorem to determine the dimension of Ker(A). Does this agree with part **a**?

The rank-nullity theorem states $dim(\mathbb{R}^3) = nullity(A) + rank(A) \rightarrow 3 = nullity(A) + 2 \rightarrow dim(ker(A)) = 1$. Yes, because in part b. we found a basis for the kernel which consist of one element.

Problem 5 (10pts/ 2pts each) Given the matrices

$$A = \begin{bmatrix} 5 & 4 \\ -3 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 1 \\ -2 & 1 & 4 \\ 2 & 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 0 & -3 & 1 \\ 1 & 1 & -2 \\ 0 & 5 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 0 & -3 & 1 & 0 \\ 1 & 0 & -1 & 2 \\ 0 & 0 & 1 & 5 \\ 3 & -1 & 0 & 2 \end{bmatrix}$$

calculate the following determinants:

a. $\det(A) = 17$

b. $\det(B) = 19$

c. $\det(C) = 8$

d. $\det(BC) = \det(B)\det(C) = 19 \times 8 = 152$ (Hint: Can use results from b. and c.)

e. $\det(CB) = \det(C)\det(B) = \det(B)\det(C) = 19 \times 8 = 152$ (Hint: Can use results from b. and c.)

f. (not graded) $\det(D) = -52$

You can verify answers using Matlab.

Problem 6 (20pts–5pts/each)

a. Find the determinant of the following matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 2 \\ 2 & 0 & 1 \end{bmatrix}$$

$\det(A) = 0$

c. Find a basis for the Image of A.

$$rref(A) = \begin{bmatrix} 1 & 0 & .5 \\ 0 & 1 & .25 \\ 0 & 0 & 0 \end{bmatrix}$$

The pivot points are in the first two columns so we take the first two columns of the matrix A as the basis

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} \right\}$$

d. Find a basis for the Kernel of A.

Using the row canonical form we solve $Ax = 0$

$$x_1 + .5x_3 = 0 \rightarrow x_1 = -.5x_3$$
$$x_2 + .25x_3 = 0 \rightarrow x_2 = -.25x_3$$

A solution vector has the form

$$\begin{bmatrix} -.5x_3 \\ -.25x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -.5 \\ -.25 \\ 1 \end{bmatrix}$$

and so a basis is

$$\left\{ \begin{bmatrix} -.5 \\ -.25 \\ 1 \end{bmatrix} \right\}$$

e. Does A^{-1} exist?

No, because the matrix A is singular.

Problem 7 (20 pts- 5pts each) Consider the following system

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 1 \\ x_1 + x_3 &= 0 \\ 2x_1 + x_2 + 3x_3 &= 1 \\ x_1 - 5x_2 &= 0 \end{aligned}$$

a. There are more equations than unknowns. More equations than unknowns always implies a linearly dependent set if not inconsistent. Apply Gaussian elimination steps (towards row echelon form) until the system reduces to 3 equations. Note that the remaining equations in echelon form are always linearly independent. No need to fully reduce system.

The third equation is the sum of the first two, so I remove it from the system of equations. This is a short cut path.

b. Put the partially reduced system (3 equations) in matrix multiplication form $A\vec{x} = \vec{b}$. What is A, $\vec{x}$, and $\vec{b}$?

Credit given for a 3×3 A matrix.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & -5 & 0 \end{bmatrix}$$

with corresponding $\vec{b}$ vector

$$\vec{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Answers can vary.

c. Calculate A^{-1} by reducing the left side of the augmented matrix $[A|I]$ to row canonical form, where I is the identity matrix. Solution can be verified using the function inv in Matlab.

Credit given if the augmented matrix $[A|I]$ is transformed into $[I|A^{-1}]$ using elementary row operations. The inverse of A is

$$A^{-1} = \begin{bmatrix} -1.25 & 2.5 & -.25 \\ -.25 & .5 & -.25 \\ 1.25 & -1.5 & .25 \end{bmatrix}$$

Answers can vary. The dim(Ker(A))=0, since the matrix is invertible this implies one-to-one mapping, meaning no non-zero vectors map to zero.

d. Find the solution $\vec{x}$ using the calculated inverse. Show work.

$$\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} -1.25 & 2.5 & -.25 \\ -.25 & .5 & -.25 \\ 1.25 & -1.5 & .25 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1.25 \\ -.25 \\ 1.25 \end{bmatrix}.$$

e. (not graded) Verify your answer by applying the Matlab function rref on the augmented matrix of the original system to show that the answers are the same. Write down the output.

$$\text{rref}\left(\begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 3 & 1 \\ 1 & -5 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 & -1.25 \\ 0 & 1 & 0 & -.25 \\ 0 & 0 & 1 & 1.25 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

***10pts for submission**