AMS10 HW5-Grading Rubric

*On all problems you can use Matlab to verify your answer but you must show work unless otherwise indicated.

Problem 1 (25pts) Consider the following set of vectors

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \right\} = \{v_1, v_2\}.$$

$Span(\{v_1, v_2\})$ is the set of all linear combinations of $\{v_1, v_2\}$:

$$k_1 v_1 + k_2 v_2 : k_1, k_2 \in \mathbb{R}.$$

Let $\vec{b} = [a, b, c]^T \in Span(S)$. Note, we can rewrite this as

$$[v_1, v_2] \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

a. (5pts for effort) Put the augmented matrix $[A|\vec{b}]$ in row canonical form, carrying through variables a, b, c.

$$\left[\begin{array}{cc|c} 1 & -1 & a \\ 0 & 2 & b \\ 0 & 0 & c \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & b/2 + a \\ 0 & 1 & b/2 \\ 0 & 0 & c \end{array}\right]$$

b. (5pts for effort) What constraints if any are placed on the variables a, b, c.
If $c \neq 0$, then the system is inconsistent and has no solution.

c. (5pts for effort) Define the set of vectors that is not in the span of $\{v_1, v_2\}$

$$\left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : c \neq 0 \right\}$$

d. (10pts) Define the set of vectors in the span of $\{v_1, v_2\}$

$$\left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : c = 0 \right\}$$

Problem 2 (25pts) Consider the following vectors

$$v_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ -2 \\ -4 \end{bmatrix}, v_3 = \begin{bmatrix} -2 \\ 6 \\ 14 \end{bmatrix}.$$

a. (5pts for effort) What is the rank of matrix $A = [v_1, v_2, v_3]$? What does this say about the dimension of $colsp(A)$ (also the image of A and span(v_1, v_2, v_3)?

$$rref(A) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

The rank(A)= 2, since the echelon form has two pivot points. This means there are maximum two linearly independent column vectors and so the dim(colsp(A))=2, which is also the dim(Im(A)) and equivalently the dim(span($\{v_1, v_2, v_3\}$)).

b. (15pts) Note that solving $A\vec{x} = \vec{0}$, gives the linear combination of the vectors v_1, v_2, v_3 that maps to the zero vector. Hence, it gives vectors $\vec{x}$ in the nullspace of A. Use the reduced matrix of A (you may use rref([A|0]) in Matlab) to find a vector $\vec{x}$ that satisfies the system of equations and verify it is in the null space of A by showing $A\vec{x} = 0$. The reduced augmented matrix is

$$rref([A, 0]) = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

A solution is given by

$$\textbf{(5pts)} x_3 : \text{free variable} \tag{145}$$
$$x_2 = 3x_3 \tag{146}$$
$$x_1 = -2x_3 \tag{147}$$

and, hence, all vectors of the form

$$\textbf{(5pts)} \vec{x} = \begin{bmatrix} -2x_3 \\ 3x_3 \\ x_3 \end{bmatrix}.$$

Choosing $\vec{x} = [-2, 3, 1]^T$, calculate $A\vec{x}$

$$\textbf{(5pts)} \begin{bmatrix} -1 & 0 & -2 \\ 0 & -2 & 6 \\ 1 & -4 & 14 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

c. (5pts for effort) Use the Rank-Nulliity theorem to determine the dimension of Ker(A). Does this agree with part b?

The rank-nullity theorem states $dim(\mathbb{R}^3) = nullity(A) + rank(A) \to 3 = nullity(A) + 2 \to dim(ker(A)) = 1$. Yes, because in part b. we found a basis for the kernel which consist of one element.

Problem 3 (25pts) Let matrix A be composed of

$$v_1 = \begin{bmatrix} 9 \\ -4 \\ 5 \\ 5 \end{bmatrix}, v_2 = \begin{bmatrix} -19 \\ 2 \\ -19 \\ -2 \end{bmatrix}, v_3 = \begin{bmatrix} -34 \\ 28 \\ -2 \\ -36 \end{bmatrix}, v_4 = \begin{bmatrix} 1 \\ 6 \\ 9 \\ -8 \end{bmatrix}.$$

In this problem you may use Matlab to reduce matrices to row canonical form.

a. (5pts for effort) Reduce the matrix $A = [v_1, v_2, v_3, v_4]$ to row canonical form. Are the vectors v_1, v_2, v_3, v_4 linearly dependent or independent?

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & -8 & -2 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The vectors are linearly dependent. The rank of the matrix is two but there are four vectors. Rank is equal to the maximum number of linearly independent vectors.

b. (5pts for effort) Reduce the matrix $A = [v_1, v_2]$ to row canonical form. Are the vectors v_1, v_2 linearly dependent or independent? How does the row canonical form compare to the first two columns for the reduced matrix in part a.

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The vectors are linearly independent since the matrix has rank 2 and the number of column vectors are two. The row canonical form is identical to the first two columns.

c. (10 pts) Reduce the matrix $A = [v_1, v_2, v_3]$ to row canonical form. Are the vectors v_1, v_2, v_3 linearly dependent or independent? How does the row canonical form compare to the first three columns for the reduced matrix in part a.

$$(\textbf{5pts})\text{rref}(A) = \begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The vectors are linearly dependent (**5pts**) since the matrix is rank 2 and there are three vectors. The row canonical form is identical to the first three columns.

d. Reduce the matrix $A = [v_3, v_4]$ to row canonical form. Are the vectors v_3, v_4 linearly dependent or independent?

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The vectors are linearly independent since the matrix has rank 2 and the number of column vectors are two.

e. (5pts) Will any subset of three vectors always be linearly dependent?

Any subset of three vectors will be linearly dependent. The dimension of the span of the vectors is two.

f. Give a basis for the span$\{v_1, v_2, v_3, v_4\}$.

A basis can be any set of two linearly independent vectors from the set. A basis can be $\{v_1, v_2\}$ or $\{v_3, v_4\}$.

Problem 4 (25pts) Consider the following set of vectors

$$v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -2 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ -3 \\ -2 \end{bmatrix}, v_4 = \begin{bmatrix} -2 \\ -2 \\ 2 \\ -3 \\ 8 \end{bmatrix}.$$

a. (10pts) Find the dimension of the subspace spanned by the vectors

One way to determine the dimension of the subspace is to find the rank of a matrix composed of the vectors. We construct a matrix composed of the column vectors and find the echelon form

$$\text{rref}([v_1, v_2, v_3, v_4]) = \begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The matrix has rank(A)= 3 and so the dim(span($\{v_1, v_2, v_3, v_4\}$))=3 (**10pts**).

b. (10pts for effort) Find a basis for the subspace spanned by the vectors

We can construct a matrix where the row vectors are the given vectors and reduce to row canonical form:

$$\text{rref}(\left[v_1^T; v_2^T; v_3^T; v_4^T\right]) = \begin{bmatrix} 1 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & -1.5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

then a basis would consist of the nonzero rows

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1.5 \\ 0 \end{bmatrix} \right\}.$$

Equivalently one can take the vectors corresponding to the pivot points of the reduced matrix composed of the column vectors

$$\text{rref}([v_1, v_2, v_3, v_4]) = \begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which tells us $\{v_1, v_2, v_3\}$ is a basis.

c. (5pts for effort) What is the rank of the matrix $A = [v_1, v_2, v_3, v_4]$?

We found in part a. that rank(A)=3.