AMS212a HW5

S-L eigenvalue problem
1. The motion of a piano string of length \( L \), fixed at each end (hint: this gives boundary conditions for \( u(0,t) = u(L,t) \)), is governed by the wave equation

\[
u_{tt} - c^2 u_{xx} = Q(x,t)\]

where \( c \) is the wave speed on the string and \( Q(x,t) \) is the forcing term.

The string \( L \) is struck, at \( t = 0 \), by a flat hammer of width \( 2d \) centered at a point \( x_0 \), having a velocity \( v_0 \). Find the resulting motion of the string \( u(x,t) \). You can assume \( Q = 0 \) and that the effect of the hammer is to produce an initial velocity distribution

\[
u_t(x,0) = \begin{cases} v_0 & \text{for } |x - x_0| < d \\ 0 & \text{for } |x - x_0| > d. \end{cases} \quad (49)\]

Additionally, assume that \( d < x_0 < L - d \). That is the length of the string is bigger than the width of the hammer.

Green’s function from transforms
2. (a) Find the Green’s function that solves the 1D wave equation

\[
u_{tt} - c^2 u_{xx} = \delta(x) \delta(t), \quad x \in (-\infty, \infty) \quad (50)\]

\[
u(t = 0, x) = 0 \quad (51)\]

\[
u_t(t = 0, x) = 0 \quad (52)\]

You may use the following known transforms: Laplace transform

\[\mathcal{L}[H(t - \tau)] = \frac{1}{s} e^{-\tau s},\]

where \( H(t - \tau) \) is the Heaviside step function and the Fourier transform

\[\mathcal{F}[e^{-\alpha|x|}] = \frac{2\alpha}{\alpha^2 + k^2}.\]

(b) Use the Green’s function to solve the wave equation with sinusoidal pumping

\[
u_{tt} - c^2 u_{xx} = \delta(x) \sin(\omega_0 t) \quad (53)\]

\[
u(t = 0, x) = 0 \quad (54)\]

\[
u_t(t = 0, x) = 0 \quad (55)\]

and show that the solution coincides with the solution given in class

\[\nu(t, x) = \frac{1}{2c\omega_0} \left[ 1 - \cos \left( \omega_0 \left( t - \frac{|x|}{c} \right) \right) \right] H \left( t - \frac{|x|}{c} \right).\]

Laplacian in 2D
3. Consider the general 2D problem

\[\nabla^2 u = f(x,y)\]

on a given finite domain \( \Omega \) with Dirichlet boundary condition

\[\nu(x,y)|_{\partial\Omega} = g(x,y).\]
In class we derived the following general expression relating the associated Green’s function to the solution
\[
    u(x, y) = \int_{\partial \Omega} (u(\xi, \eta) \nabla G(x, y|\xi, \eta) - G(x, y|\xi, \eta) \nabla u(\xi, \eta)) \cdot \hat{n} \, dS + \int_{\Omega} \nabla^{2} u(\xi, \eta) \, dV
\]
Recall, we did this applying the divergence theorem.

We want to solve the problem below on a rectangular domain
\[
    \nabla^{2} u = f(x, y)
\]
\[
    u(0, y) = a(y)
\]
\[
    u_{x}(L, y) = b(y)
\]
\[
    u_{y}(x, 0) = c(x)
\]
\[
    u_{y}(x, H) = d(x)
\]
Suppose you know the associated Green’s function, given by
\[
    \nabla^{2} G = \delta(x - \xi, y - \eta)
\]
where \(G\) vanishes on \(x = 0\) and its normal derivative vanishes on \(x = L, y = 0, H\). Using the solution expression above write down what the general solution simplifies to as a function of the forcing function, the boundary conditions and known rectangular domain. (Be careful with signs, recall \(\hat{n}\) is unit normal on surface)

**Method of images**

4. Use method of images to solve
\[
    \nabla^{2} G = \delta(\vec{x} - \vec{\xi})
\]
where \(\vec{x} \in \mathbb{R}^{2}\) with \(G = 0\) on the appropriate boundaries using the free space Green’s function
\[
    K(\vec{x}|\vec{\xi}) = \frac{1}{2\pi} \log |\vec{x} - \vec{\xi}|.
\]
(a) In the first quadrant of two-dimensional space \(0 < x < \infty, 0 < y < \infty\)
\[
    \nabla^{2} G = \delta(x - \xi) \delta(y - \eta), 0 < x, y < \infty
\]
\[
    G(x = 0) = G(y = 0) = 0
\]
(Hint: symmetrically place delta functions and solve for appropriate coefficients by applying boundary conditions)
(b) Upper half plane semi-circle
\[
    \nabla^{2} G = \delta(x - \xi) \delta(y - \eta), 0 < r < a, 0 < \theta < \pi
\]
\[
    G(r = a) = 0, 0 < \theta < \pi
\]
\[
    G(\theta = 0) = G(\theta = \pi) = 0, 0 < r < a
\]
Recall from class that the Green’s solution for
\[
    \nabla^{2} G = \delta(\vec{x} - \vec{\xi}), 0 \leq r < a, 0 \leq \theta < 2\pi
\]
\[
    G(a, \theta) = 0
\]
was

\[ G(\vec{x}|\vec{\xi}) = \frac{1}{4\pi} \log \left( \frac{a^2 \left( r^2 + |\vec{\xi}|^2 - 2r|\vec{\xi}| \cos(\theta - \theta_{\vec{\xi}}) \right)}{a^4 + r^2|\vec{\xi}|^2 - 2r|\vec{\xi}|a^2 \cos(\theta - \theta_{\vec{\xi}})} \right). \]

Hint: Use this solution to solve the problem. Any linear combination of this solution or rotations will satisfy \( G(a, \theta) = 0 \). Recall method of images to solve Green’s function on a semi-infinite plane and apply similar principle.