AMS10 HW5- Grading Rubric

Problem 1. (28pts–4pts/each)

a. Eigenvalues of the matrix

$$A = \begin{bmatrix} 5 & 4 \\ 3 & 1 \end{bmatrix}$$

in Problem 1 can be calculated by solving the system $\det(sI - A) = 0$. Solve for s:

$$\det\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 5 & 4 \\ 3 & 1 \end{bmatrix}\right) = 0.$$

You may use the quadratic formula. The general quadratic equation $ax^2 + bx + c = 0$ has the following solution

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

$$\det\left(\begin{bmatrix} s-5 & -4 \\ -3 & s-1 \end{bmatrix}\right) = (s-5)(s-1) - 12 = s^2 - 6s - 7.$$

$$\lambda_{1,2} = \frac{6 \pm \sqrt{6^2 - 4(-7)}}{2} \rightarrow \lambda_1 = 7, \lambda_2 = -1$$

b. Find the determinant of the matrix and verify that $\det(A) = \lambda_1\lambda_2$

$$\det(A) = 5(1) - 4(3) = 5 - 12 = -7$$

c. Find a basis for the Image of A.

$$rref(A) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\left\{\begin{bmatrix} 5 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \end{bmatrix}\right\}$$

d. Find a basis for the Kernel of A.

The matrix is full rank and there is no zero eigenvalue so the Kernel of A is $\left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$

e. Does A^{-1} exist?

Yes, the matrix is full rank and invertible and so A^{-1} exists.

f. For each of the eigenvalues calculate the matrix $M_i = A - \lambda_i I$

$$M_1 = A - 7I = \begin{bmatrix} -2 & 4 \\ 3 & -6 \end{bmatrix}$$

$$M_2 = A + I = \begin{bmatrix} 6 & 4 \\ 3 & 2 \end{bmatrix}$$

g. Find the basis for the Kernel of each of the matrices M_1 and M_2 found in part f. These are the eigenvectors.

$$rref(M_1) = \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \rightarrow \left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$$

$$rref(M_2) = \begin{bmatrix} 1 & 2/3 \\ 0 & 0 \end{bmatrix} \to \left\{ \begin{bmatrix} -2/3 \\ 1 \end{bmatrix} \right\}$$

Problem 2 (12 pts-4pts/each) Consider the following matrix

$$A = \begin{bmatrix} -1 & 2 & 4 \\ 0 & 5 & 2 \\ 2 & -1 & 9 \end{bmatrix}$$

a. Calculate the eigenvalues of A.

Eigenvalues are $\lambda_1 = -1.6217, \lambda_2 = 9.48671, \lambda_3 = 5.1346$ found by using eig(A) in Matlab.

b. Calculate the eigenvalues of the row canonical form of A. How do the eigenvalues compare to those of A?

$$rref(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which has eigenvalue $\lambda = 1$ with algebraic multiplicity 3. The eigenvalues of the matrix is row canonical form are not equal to the eigenvalues of the original matrix.

c. Calculate the eigenvalues of A^T. How do the eigenvalues compare to those of A?

The eigenvalues of A^T are equal to the eigenvalues of A.

$*$ For this problem you may use Matlab

Problem 3 (28 pts-4pts/each) Consider the following matrix

$$A = \begin{bmatrix} -1 & 2 \\ 0 & 5 \end{bmatrix}$$

a. What method can be used to show that this matrix is diagonalizable?

A square $n \times n$ matrix is diagonalizable if and only if it has n linearly independent eigenvectors. If the eigenvalues are distinct then we know that the matrix is diagonalizable because the corresponding eigenvectors are guaranteed to be linearly independent.

b. Calculate by hand the eigenvalues and associated eigenvectors.

The matrix is an upper triangular matrix and so the characteristic polynomial is

$$p(s) = (s+1)(s-5),$$

which has roots and, hence, eigenvalues $\lambda = -1$ and $\lambda = 5$. An eigenvector for $\lambda = -1$ is found by solving

$$(A - (-1)I)x = 0 \implies \begin{bmatrix} 0 & 2 \\ 0 & 6 \end{bmatrix} x = \vec{0} \implies x_2 = 0$$

and x_1 is a free variable. So an eigenvector is

$$x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

An eigenvector for $\lambda = 5$ is found by solving

$$(A - (5)I)x = 0 \implies \begin{bmatrix} -6 & 2 \\ 0 & 0 \end{bmatrix} x = \vec{0} \implies 6x_1 = 2x_2$$

and x_2 is a free variable. So an eigenvector is

$$x = \begin{bmatrix} 1/3 \\ 1 \end{bmatrix}.$$

c. Find the matrix P such that $A = PDP^{-1}$.
P is constructed from the eigenvectors.

$$P = \begin{bmatrix} 1 & 1/3 \\ 0 & 1 \end{bmatrix}$$

d. Derive the relation $D = P^{-1}AP$ from $A = PDP^{-1}$. (Hint: $PP^{-1} = P^{-1}P = I$)

$$\begin{aligned} A &= PDP^{-1} \\ P^{-1}(A &= PDP^{-1})P \\ P^{-1}AP &= P^{-1}PDP^{-1}P \\ P^{-1}AP &= IDI \\ P^{-1}AP &= D \end{aligned}$$

e. Use the matrix P to compute by hand the diagonal matrix $D = P^{-1}AP$ and verify that the elements along the diagonal are indeed the eigenvalues of the matrix A.

$$P^{-1} = \frac{1}{det(P)} \begin{bmatrix} 1 & -1/3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1/3 \\ 0 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & -1/3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1/3 \\ 0 & 1 \end{bmatrix} = . \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}$$

f. Using the relation $A = PDP^{-1}$, compute by hand A^3 by computing $(PDP^{-1})^3$. (Hint: $PP^{-1} = P^{-1}P = I$)

$$\begin{aligned} A^3 &= (PDP^{-1})(PDP^{-1})(PDP^{-1}) \\ A^3 &= PD(P^{-1}P)D(P^{-1}P)DP^{-1} \\ A^3 &= PDIDIDP^{-1} \\ A^3 &= PD^3P^{-1} \end{aligned} \tag{1}$$

g. Using the results from **f** determine the eigenvalues of A^3 without deriving the characteristic polynomial.

$$A^3 = PD^3P^{-1} = P \begin{bmatrix} (-1)^3 & 0 \\ 0 & 5^3 \end{bmatrix} P^{-1}$$

and so by inspection A^3 is diagonalizable and its eigenvalues are found in the diagonal matrix, so $\lambda_1 = -1, \lambda_2 = 5^3$.

Problem 4 (16 pts-4pts/each) Consider the following matrix

$$A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 1 \\ 3 & 0 & 1 \end{bmatrix}$$

a. Find the characteristic polynomial $p(s)$ of the matrix A.

$$det(sI - A) = det\left(\begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 1 \\ 3 & 0 & 1 \end{bmatrix}\right) = det\left(\begin{bmatrix} s+2 & 0 & 0 \\ 0 & s-3 & -1 \\ -3 & 0 & s-1 \end{bmatrix}\right)$$

$$det(sI - A) = (s+2)(s-3)(s-1) = s^3 - 2s^2 - 5s + 6$$

b. Show that the matrix A satisfies the characteristic polynomial. That is $p(A)$ gives the 3×3 zero matrix. You may use Matlab to compute after setting up the problem.

$$p(A) = A^3 - 2A^2 - 5A + 6I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

c. Compute the eigenvalues of A. You may use Matlab here.
The eigenvalues are $\lambda_1 = -2, \lambda_2 = 3, \lambda_3 = 1$

d. For each of the eigenvalues $\lambda_1, \lambda_2, \lambda_3$, compute the associated eigenvectors. You may use Matlab. Show that v_i is in the null space of $A - \lambda_i I$ by computing $(A - \lambda_i I)v_i$, where v_i is the eigenvector associated with the eigenvalue λ_i. Show that v_i is invariant under the transformation A by computing Av_i.

Eigenspace of $\lambda = 3$ is $span\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$ and $(A - 3 * eye(3)) * [0; 1; 0] = [0; 0; 0]$

Eigenspace of $\lambda = 1$ is $span\left\{\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}\right\}$ and $(A - 1 * eye(3)) * [0; -1; 2] = [0; 0; 0]$

Eigenspace of $\lambda = -2$ is $span\left\{\begin{bmatrix} 5 \\ 1 \\ -5 \end{bmatrix}\right\}$ and $(A + 2 * eye(3)) * [5; 1; -5] = [0; 0; 0]$

Problem 5 (12 pts-4pts/each) Consider the following matrix

$$A = \begin{bmatrix} -5 & 2 & -1 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

a. Compute the eigenvalues by hand. (Hint: Use the transpose A^T to compute the eigenvalues using the method discussed in class. This eases computation). How do the eigenvalues compare to the diagonal entries of the matrix? Note, this property always holds true for an upper triangular matrix such as A (and a lower triangular matrix such as A^T).

This is an upper triangular matrix and so the eigenvalues are those on the diagonal: $\lambda_1 = -5, \lambda_2 = 1, \lambda_3 = 1, \lambda_4 = 3$.

b. For any repeated eigenvalues, find the eigenvectors by hand. Are the eigenvectors linearly independent? The number of times the eigenvalue is repeated is its algebraic multiplicity. What is the algebraic multiplicity for repeated eigenvalues? The geometric multiplicity is given by the maximum number of linearly independent associated eigenvectors. What is the geometric multiplicity for repeated eigenvalues?

$\lambda = 1$ is the only repeated eigenvalue with an algebraic multiplicity 2.

$$(A - (1)I) = \begin{bmatrix} -6 & 2 & -1 & 3 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

and so $x_4 = 0$, while x_2 and x_3 are free variables. Furthermore $x_1 = (2/6)x_2 - (1/6)x_3$. An eigenvector is in the span of the following set of linearly independent vectors

$$\left\{ \begin{bmatrix} 2/6 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/6 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

and so $\lambda = 1$ has geometric multiplicity equal to 2.

c. Consider the matrix

$$B = \begin{bmatrix} -5 & 2 & -1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

What is the algebraic multiplicity and geometric multiplicity for any repeated eigenvalues? You may use Matlab. Is the matrix B diagonalizable?

Using Matlab, we find that $\lambda = 1$ has algebraic multiplicity of 2 but only a geometric multiplicity of one. Matrix B has only three linearly independent eigenvectors. Matrix B is not diagonalizable.

*4pts for submission