AMS10 HW6

Problem 1 In two dimensions, the standard rotation matrix has the following form

$$A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix},$$

which rotates column vectors by an angle θ.

a. For $\theta = \pi/4$ compute by hand the eigenvalues and eigenvectors. Plot the vector $\vec{x} = [1, 0]^T$ and the resulting vector $A\vec{x}$ after transformation.

b. For $\theta = \pi/2$ compute by hand the eigenvalues and eigenvectors. Plot the vector $\vec{x} = [1, 0]^T$ and the resulting vector $A\vec{x}$ after transformation.

c. Is any real vector direction invariant under this transformation (i.e., unchanging when transformed by A for arbitrary θ)?

Problem 2 Let

$$\vec{u} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \ \vec{v} = \begin{bmatrix} -3 \\ 2 \\ 2 \end{bmatrix}.$$

Compute the following quantities.

a. $\vec{u}^T\vec{v}$

b. $\vec{v}^T\vec{u}$

c.. $\left(\frac{\vec{u}^T\vec{u}}{\vec{v}^T\vec{u}}\right)\vec{u}$

d.. $||\vec{u} - \vec{v}||$

Problem 3 Use SVD to compute the pseudoinverse of A to solve the system $A\vec{x} = \vec{b}$ for

$$A = \begin{bmatrix} 0 & 0 & 4 \\ 2 & 3 & -1 \\ 0 & 1 & 8 \\ 2 & 4 & 7 \\ 4 & 7 & 6 \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} -1 \\ -3 \\ 0 \\ -3 \\ -6 \end{bmatrix}. \tag{1}$$

You can use Matlab to compute everything in this problem.

a. SVD has the following form,

$$A = U\Sigma V^*. \tag{2}$$

The asterisk $*$ denotes taking the transpose of the matrix and the complex conjugate of any complex entries. However, we are dealing with only real matrices here so for the purpose of this example A^* can be replaced with A^T. Compute the matrices U and V.

1. Use Matlab to get the eigenvectors of AA^* and A^*A, which compose the matrices $U \in \mathbb{C}^{5\times5}$ and $V^* \in \mathbb{C}^{3\times3}$, respectively. For example $[U, D] = eig(AA^T)$ and $[V, D] = eig(A^TA)$.

2. Compute Σ using the following formula $\Sigma = U^TAV$ with the newly found matrices U and V

3. Verify that the eigenvalues of AA^* or A^*A (check that they are equal) are the squares of the diagonal elements of $\Sigma \in \mathbb{R}^{5\times3}$. No need to prove this verification. This is just a checkpoint.

b. Compute the pseudo inverse of A as $A^+ = V\Sigma^+ U^*$ where the pseudoinverse $\Sigma^+ \in \mathbb{R}^{3\times 5}$ is the matrix where the non-zero entries are replaced with their reciprocal $a \implies 1/a$ and then the matrix is transposed.
c. Compute $\vec{x} = A^+\vec{b}$ and confirm your solution by computing $A\vec{x}$. Check your solution with the pinv Matlab command (Ex: $x = pinv(A) * \vec{b}$). No need to prove Matlab work. This function in Matlab computes the pseudo inverse. There is also a command *svd* that gives you U, V and Σ.

Problem 4 Determine if the following pairs of vectors are orthogonal.

$$\left\{\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2.5 \end{bmatrix}\right\}, \left\{\begin{bmatrix} -3 \\ 7 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -8 \\ 25 \\ -7 \end{bmatrix}\right\}, \left\{\begin{bmatrix} 13 \\ -3 \\ -7 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right\}$$

Problem 5 Let $W = colsp(A)$, where

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 1 \end{bmatrix}.$$

Find $W^\perp$, the orthogonal complement of W.

Problem 6 Let

$$\vec{u} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix},$$

and $W = span\{\vec{u}\}$. What is the dimension of $W^\perp$, the orthogonal complement of W?

Problem 7 Let A be a 7×5 matrix. What is the smallest possible dimension of $[colsp(A)]^\perp$?

Problem 8 Let

$$\vec{u_1} = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}, \vec{u_2} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \vec{u_3} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, \text{ and } \vec{x} = \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix}.$$

Express $\vec{x}$ as a linear combination of $\{\vec{u_1}, \vec{u_2}, \vec{u_3}\}$, that is $\vec{x} = c_1\vec{u_1} + c_2\vec{u_2} + c_3\vec{u_3}$. Compute the coefficients using the fact that the set of vectors are orthogonal.

Problem 9 Let

$$W = span\left\{\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\right\} \text{ and } \vec{y} = \begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix}$$

Find $\text{proj}_W\vec{y}$.