AMS10 HW6 Solutions

Problem 1 (30 pts-10pts/each) In two dimensions, the standard rotation matrix has the following form

$$A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix},$$

which rotates column vectors by an angle θ.

We begin by finding the eigenvalues and eigenfunctions in terms of θ.

$$p(s) = \det(sI - A) = \det\left(\begin{bmatrix} s - \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & s - \cos(\theta) \end{bmatrix} \right) = (s - \cos(\theta))^2 + \sin^2(\theta)$$

$$p(\lambda) = 0 \implies \lambda^2 - 2\lambda\cos(\theta) + 1 = 0 \implies \lambda_1 = \cos(\theta) + i\sin(\theta), \lambda_2 = \cos(\theta) - i\sin(\theta)$$

$$(A - \lambda_1 I)\vec{v}_1 = \begin{bmatrix} -i\sin(\theta) & -\sin(\theta) \\ \sin(\theta) & -i\sin(\theta) \end{bmatrix} \vec{v}_1 = 0 \implies \vec{v}_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$(A - \lambda_2 I)\vec{v}_2 = \begin{bmatrix} i\sin(\theta) & -\sin(\theta) \\ \sin(\theta) & i\sin(\theta) \end{bmatrix} \vec{v}_2 = 0 \implies \vec{v}_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

Note that the eigenvectors are the same for any θ such that $\sin(\theta) \neq 0$, otherwise all vectors are in the eigenspace.

a. For $\theta = \pi/4$ compute by hand the eigenvalues and eigenvectors. Plot the vector $x = [1, 0]^T$ and the resulting vector Ax after transformation.

$\lambda_1 = \cos(\pi/4) + i\sin(\pi/4) = 1/\sqrt{2} + i/\sqrt{2}, \quad \lambda_2 = \cos(\pi/4) - i\sin(\pi/4) = 1/\sqrt{2} - i/\sqrt{2}$

$$A\vec{x} = A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

b. For $\theta = \pi/2$ compute by hand the eigenvalues and eigenvectors. Plot the vector $x = [1, 0]^T$ and the resulting vector Ax after transformation.

$\lambda_1 = \cos(\pi/2) + i\sin(\pi/2) = i, \quad \lambda_2 = \cos(\pi/2) - i\sin(\pi/2) = -i$

$$A\vec{x} = A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

c. Is any real vector invariant under this transformation?

For $\theta \neq 0$, no, because the eigenvectors are complex. There are no real eigenvectors and so there does not exist a real vector that is invariant under a rotational transformation, except for $\vec{0}$. For $\theta = 0, 2\pi, 4\pi, \ldots$, yes, for all vectors.

Problem 2 (20pts-5pts/each) Let

$$u = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \ v = \begin{bmatrix} -3 \\ 2 \\ 2 \end{bmatrix}.$$

Compute the following quantities.

a. $u^T v$

$$[1\,3\ -2]\begin{bmatrix} -3 \\ 2 \\ 2 \end{bmatrix} = 1\cdot(-3) + 3\cdot(2) - 2\cdot(2) = -1$$

b. $v^T u$

$$v^T u = u^T v = -1$$

c.. $\left(\frac{u^T u}{v^T u}\right) u$

$$[1\,3\ -2]\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} = 1\cdot(1) + 3\cdot(9) - 2\cdot(-2) = 14$$

$$\left(\frac{u^T u}{v^T u}\right) u = \left(\frac{14}{-1}\right) u = -14\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -14 \\ -42 \\ 28 \end{bmatrix}$$

d. $||u - v||$

$$\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} - \begin{bmatrix} -3 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ -4 \end{bmatrix}$$

$$||u - v|| = \sqrt{^2 + 1^1 + (-4)^2} = \sqrt{33}$$

Problem 3 (30pts-10pts/each) Use SVD to compute the pseudoinverse of A to solve the system $A\vec{x} = \vec{b}$ for

$$A = \begin{bmatrix} 0 & 0 & 4 \\ 2 & 3 & -1 \\ 0 & 1 & 8 \\ 2 & 4 & 7 \\ 4 & 7 & 6 \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} -1 \\ -3 \\ 0 \\ -3 \\ -6 \end{bmatrix}. \tag{3}$$

a. SVD has the following form,

$$A = U\Sigma V^* \tag{4}$$

Use Matlab to get the eigenvectors of AA^* and A^*A which compose the matrices $U \in \mathbb{C}^{5\times 5}$ and $V^* \in \mathbb{C}^{3\times 3}$, respectively. The eigenvalues of AA^* or A^*A (check that they are equal) are the *squares* of the diagonal elements of $\Sigma \in \mathbb{R}^{5\times 3}$, and the lower $5 - 3 = 2$ rows of Σ are filled with zeros.

$$U = \begin{bmatrix} 0.2227 & -0.3392 & 0.9140 & 0.0000 & -0.0000 \\ 0.0787 & 0.5422 & 0.1821 & 0.8115 & -0.0904 \\ 0.4787 & -0.5736 & -0.3295 & 0.3606 & -0.4509 \\ 0.5573 & -0.0313 & -0.1474 & 0.0904 & 0.8115 \\ 0.6360 & 0.5109 & 0.0347 & -0.4509 & -0.3606 \end{bmatrix}, \qquad \Sigma = \begin{bmatrix} 14.8996 & 0 & 0 \\ 0 & 6.5530 & 0 \\ 0 & 0 & 0.2458 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$V^* = \begin{bmatrix} 0.2561 & 0.4964 & 0.8295 \\ 0.4678 & 0.6873 & -0.5557 \\ 0.8459 & -0.5303 & 0.0562 \end{bmatrix}$$

b. Compute the pseudoinverse of A as $A^+ = V\Sigma^+ U^*$ where the pseudoinverse of Σ has diagonal elements that the inverses of those in Σ. Check your solution with the pinv command.

$$\Sigma^+ = \begin{bmatrix} 0.0671 & 0 & 0 & 0 & 0 \\ 0 & 0.1526 & 0 & 0 & 0 \\ 0 & 0 & 4.0682 & 0 & 0 \end{bmatrix}, A^+ = \begin{bmatrix} 3.1250 & 0.6667 & -1.1667 & -0.5000 & 0.1667 \\ -2.0000 & -0.3333 & 0.6667 & 0.3333 & -0.0000 \\ 0.2500 & -0.0000 & -0.0000 & 0.0000 & 0.0000 \end{bmatrix}$$

c. Compute $\vec{x} = A^+\vec{b}$ and confirm your solution by computing $A\vec{x}$.

$$\vec{x} = \begin{bmatrix} -4.6250 \\ 2.0000 \\ -0.2500 \end{bmatrix}, \qquad A\vec{x} = \vec{b} = \begin{bmatrix} -1 \\ -3 \\ 0 \\ -3 \\ -6 \end{bmatrix}$$

Problem 4 (15pts-5pts/each) Determine if the following pairs of vectors are orthogonal.

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2.5 \end{bmatrix} \right\},$$

$$[2\,1\,-2] \begin{bmatrix} 2 \\ 1 \\ 2.5 \end{bmatrix} = 2 \cdot (2) + 1 \cdot (1) - 2 \cdot (2.5) = 0 \text{ orthogonal}$$

$$\left\{ \begin{bmatrix} -3 \\ 7 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -8 \\ 25 \\ -7 \end{bmatrix} \right\},$$

$$[-3\,7\,4\,0] \begin{bmatrix} 1 \\ -8 \\ 25 \\ -7 \end{bmatrix} = -3 \cdot (1) + 7 \cdot (-8) + 4 \cdot (25) + 0 \cdot -7 = -41 \text{ not orthogonal}$$

$$\left\{ \begin{bmatrix} 13 \\ -3 \\ -7 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$[13\,-3\,-7\,4] \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 13 \cdot (0) - 3 \cdot (0) - 7 \cdot (0) + 4 \cdot 0 = 0 \text{ orthogonal}$$

Problem 5 (20pts) Let $W = colsp(A)$, where

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 1 \end{bmatrix}.$$

Find $W^\perp$, the orthogonal complement of W. We begin by taking the transpose of A and then finding the null space.

$$A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \end{bmatrix}$$

$$A^T x = 0 \implies \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 = -2x_2$$
$$x_3 = 0.$$

$$x = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$$W^\perp = span\{x\}$$

Problem 6 (20pts) Let

$$u = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix},$$

and $W = span\{u\}$. What is the dimension of $W^\perp$, the orthogonal complement of W. We use the fact that

$$W + W^\perp = 3 \implies W^\perp = 3 - 1$$

since $dim(W) = 1$.

Problem 7 (20pts) Let A be a 7×5 matrix. What is the smallest possible dimension of $[colsp(A)]^\perp$?

We use the fact that

$$W + W^\perp = 7$$

given $W = dim(colsp(A))$, since the column vectors are in $\mathbb{R}^7$. If A is a 7×5 matrix, then the maximum number of linearly independent column vectors will be 5, this means the dimension of the column space will be at most 5 as well and so $dim([colsp(A)]^\perp) \geq 2$).

Problem 8 (20pts) Let

$$u_1 = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}, \; u_2 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \; u_3 = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, \text{ and } x = \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix}.$$

Express x as a linear combination of $\{u_1, u_2, u_3\}$. For this problem we compute the coefficients such that

$$c_1 \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix}.$$

since the vectors are orthogonal it follows that

$$c_i = \frac{u_i^T x}{u_i^T u_i}$$

$$c_1 = \frac{u_1 \cdot x}{u_1 \cdot u_1} = \frac{24}{18} \tag{5}$$

$$c_2 = \frac{u_2 \cdot x}{u_2 \cdot u_2} = \frac{2}{9} \tag{6}$$

$$c_3 = \frac{u_3 \cdot x}{u_3 \cdot u_3} = \frac{10}{18} \tag{7}$$

Problem 9 (20pts) Let

$$W = span\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\} \text{ and } \vec{y} = \begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix}$$

Find $\text{proj}_W \vec{y}$.

We use the fact that the vectors are orthogonal. Let

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \text{ and } u_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

then

$$\begin{aligned} \text{proj}_W \vec{y} &= \frac{u_1 \cdot \vec{y}}{u_1 \cdot u_1} u_1 + \frac{u_2 \cdot \vec{y}}{u_2 \cdot u_2} u_2 \\ &= \frac{3}{6} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + \frac{-6}{3} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1.5 \\ -1.5 \\ 3 \end{bmatrix} \end{aligned}$$

***5pts for submission**