AMS10 HW7 Solutions

Problem 1 Let

$$A = \begin{bmatrix} 3/\sqrt{18} & 2/3 & 1/\sqrt{18} \\ -3/\sqrt{18} & 2/3 & 1/\sqrt{18} \\ 0 & -1/3 & 4/\sqrt{18} \end{bmatrix}. \tag{4}$$

a. Compute $A^T A$.

A is orthogonal, so $A^T = A^{-1}$, thus $A^T A = I$.

b. What is A^{-1}?

$$A^{-1} = A^T = \begin{bmatrix} 3\sqrt{18} & -3\sqrt{18} & 0 \\ 2/3 & 2/3 & -1/3 \\ 1\sqrt{18} & 1\sqrt{18} & 4\sqrt{18} \end{bmatrix}$$

Problem 2 Consider the system $A\vec{x} = \vec{b}$ for

$$A = \begin{bmatrix} 0 & 0 & 4 \\ 2 & 3 & -1 \\ 0 & 1 & 8 \\ 2 & 4 & 7 \\ 4 & 7 & 6 \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} -1 \\ -3 \\ 0 \\ -2 \\ -5 \end{bmatrix}. \tag{5}$$

a. Find the least-square solution.

We have

$$A\hat{x} = \vec{b}$$
$$\to (A^T A)\hat{x} = A^T\vec{b}$$
$$\to (A^T A)^{-1}(A^T A)\hat{x} = (A^T A)^{-1}A^T\vec{b}.$$

Thus, the least square solution is

$$\hat{x} = (A^T A)^{-1}A^T\vec{b}$$

with $A = \begin{bmatrix} 0 & 0 & 4 \\ 2 & 3 & -1 \\ 0 & 1 & 8 \\ 2 & 4 & 7 \\ 4 & 7 & 6 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} -1 \\ -3 \\ 0 \\ -2 \\ -5 \end{bmatrix}$.

Therefore,

$$\hat{x} = \left(\begin{bmatrix} 0 & 0 & 4 \\ 2 & 3 & -1 \\ 0 & 1 & 8 \\ 2 & 4 & 7 \\ 4 & 7 & 6 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 4 \\ 2 & 3 & -1 \\ 0 & 1 & 8 \\ 2 & 4 & 7 \\ 4 & 7 & 6 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 & 0 & 4 \\ 2 & 3 & -1 \\ 0 & 1 & 8 \\ 2 & 4 & 7 \\ 4 & 7 & 6 \end{bmatrix}^T \begin{bmatrix} -1 \\ -3 \\ 0 \\ -2 \\ -5 \end{bmatrix}$$

$$\to \hat{x} = \left(\begin{bmatrix} 0 & 2 & 0 & 2 & 4 \\ 0 & 3 & 1 & 4 & 7 \\ 4 & -1 & 8 & 7 & 6 \end{bmatrix} \begin{bmatrix} 0 & 0 & 4 \\ 2 & 3 & -1 \\ 0 & 1 & 8 \\ 2 & 4 & 7 \\ 4 & 7 & 6 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 & 2 & 0 & 2 & 4 \\ 0 & 3 & 1 & 4 & 7 \\ 4 & -1 & 8 & 7 & 6 \end{bmatrix} \begin{bmatrix} -1 \\ -3 \\ 0 \\ -2 \\ -5 \end{bmatrix}$$

$$\to \hat{x} = \left(\begin{bmatrix} 24 & 42 & 36 \\ 42 & 75 & 75 \\ 36 & 75 & 166 \end{bmatrix}\right)^{-1} \begin{bmatrix} 0 & 2 & 0 & 2 & 4 \\ 0 & 3 & 1 & 4 & 7 \\ 4 & -1 & 8 & 7 & 6 \end{bmatrix} \begin{bmatrix} -1 \\ -3 \\ 0 \\ -2 \\ -5 \end{bmatrix}$$

$$\to \hat{x} = \begin{bmatrix} 11.8490 & -7.4167 & 0.7812 \\ -7.4167 & 4.6667 & -0.5000 \\ 0.7812 & -0.5000 & 0.0625 \end{bmatrix} \begin{bmatrix} 0 & 2 & 0 & 2 & 4 \\ 0 & 3 & 1 & 4 & 7 \\ 4 & -1 & 8 & 7 & 6 \end{bmatrix} \begin{bmatrix} -1 \\ -3 \\ 0 \\ -2 \\ -5 \end{bmatrix} = \boxed{\begin{bmatrix} -4.9583 \\ 2.3333 \\ -0.2500 \end{bmatrix}}.$$

b. Let $W = colsp(A)$. Find an orthogonal basis using the Gram-Schmidt process.

We have $A = \begin{bmatrix} 0 & 0 & 4 \\ 2 & 3 & -1 \\ 0 & 1 & 8 \\ 2 & 4 & 7 \\ 4 & 7 & 6 \end{bmatrix}$. Let $\vec{x_1} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 2 \\ 4 \end{bmatrix}$, $\vec{x_2} = \begin{bmatrix} 0 \\ 3 \\ 1 \\ 4 \\ 7 \end{bmatrix}$, and $\vec{x_3} = \begin{bmatrix} 4 \\ -1 \\ 8 \\ 7 \\ 6 \end{bmatrix}$.

Let $W = colsp(A) = span\{\vec{x_1}, \vec{x_2}, \vec{x_3}\}$. Using the Gram-Schmidt process, we can find an orthogonal basis as

$$\vec{v_1} = \vec{x_1} = \boxed{\begin{bmatrix} 0 \\ 2 \\ 0 \\ 2 \\ 4 \end{bmatrix}}$$

$$\vec{v_2} = \vec{x_2} - \left(\frac{\vec{x_2} \cdot \vec{v_1}}{\vec{v_1} \cdot \vec{v_1}}\right) \vec{v_1} = \begin{bmatrix} 0 \\ 3 \\ 1 \\ 4 \\ 7 \end{bmatrix} - \left(\frac{42}{24}\right) \begin{bmatrix} 0 \\ 2 \\ 0 \\ 2 \\ 4 \end{bmatrix} = \boxed{\begin{bmatrix} 0 \\ -1/2 \\ 1 \\ 1/2 \\ 0 \end{bmatrix}}$$

$$\vec{v_3} = \vec{x_3} - \left(\frac{\vec{x_3} \cdot \vec{v_1}}{\vec{v_1} \cdot \vec{v_1}}\right) \vec{v_1} - \left(\frac{\vec{x_3} \cdot \vec{v_2}}{\vec{v_2} \cdot \vec{v_2}}\right) \vec{v_2} = \begin{bmatrix} 4 \\ -1 \\ 8 \\ 7 \\ 6 \end{bmatrix} - \left(\frac{36}{24}\right) \begin{bmatrix} 0 \\ 2 \\ 0 \\ 2 \\ 4 \end{bmatrix} - \left(\frac{12}{3/2}\right) \begin{bmatrix} 0 \\ -1/2 \\ 1 \\ 1/2 \\ 0 \end{bmatrix} = \boxed{\begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}.$$

c. Find $proj_W \vec{b}$ and show that it is equal to $A\hat{x}$, where $\hat{x}$ is the least-square solution.

We have

$$proj_W \vec{b} = \left(\frac{\vec{b} \cdot \vec{v_1}}{\vec{v_1} \cdot \vec{v_1}}\right) \vec{v_1} + \left(\frac{\vec{b} \cdot \vec{v_2}}{\vec{v_2} \cdot \vec{v_2}}\right) \vec{v_2} + \left(\frac{\vec{b} \cdot \vec{v_3}}{\vec{v_3} \cdot \vec{v_3}}\right) \vec{v_3}$$

$$\to proj_W \vec{b} = \left(-\frac{30}{24}\right) \begin{bmatrix} 0 \\ 2 \\ 0 \\ 2 \\ 4 \end{bmatrix} + \left(\frac{1/2}{3/2}\right) \begin{bmatrix} 0 \\ -1/2 \\ 1 \\ 1/2 \\ 0 \end{bmatrix} + \left(-\frac{4}{16}\right) \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \boxed{\begin{bmatrix} -1 \\ -8/3 \\ 1/3 \\ -7/3 \\ -5 \end{bmatrix}}.$$

We also have

$$A\hat{x} = \begin{bmatrix} 0 & 0 & 4 \\ 2 & 3 & -1 \\ 0 & 1 & 8 \\ 2 & 4 & 7 \\ 4 & 7 & 6 \end{bmatrix} \begin{bmatrix} -4.9583 \\ 2.3333 \\ -0.2500 \end{bmatrix} = \boxed{\begin{bmatrix} -1 \\ -2.6667 \\ 0.3333 \\ -2.3333 \\ -5 \end{bmatrix}}.$$

Therefore, the $proj_W\vec{b}$ is equal to $A\hat{x}$, where $\hat{x}$ is the least-square solution.

Problem 3 Let

$$A = \begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & 2 \\ -2 & 2 & 2 \end{bmatrix} \tag{6}$$

a. Find the eigenvectors of A

To get eigenvectors first find eigenvalues by solving $\det(A - \lambda I) = 0$,

$$\lambda_1 = \lambda_2 = 1,\ \lambda_3 = 10.$$

Then find eigenvectors in the nullspace of $(A - \lambda I)$ i.e., find $\vec{v}$ such that $(A - \lambda I)\vec{v} = \vec{0}$, i.e.,

$$\vec{v}_1 = [0.4793, 0.0409, 0.8767]^T, \quad \vec{v}_2 = [0.5708, 0.7442, -0.3468]^T, \quad \vec{v}_3 = [-0.6667, 0.6667, 0.3333]^T$$

b. Find an orthogonal set in the eigenspace for any repeated eigenvalues.

Using Gram-Schmidt Orthogonalization,

$$\text{Let } \vec{x}_1 = \vec{v}_1,$$
$$\vec{x}_2 = \vec{v}_2 - \frac{\vec{v}_2 \cdot \vec{x}_1}{\vec{x}_1 \cdot \vec{x}_1}\vec{x}_1 = [-0.5708, 0.7442, -0.3468]^T$$

Check: $\vec{x}_1 \cdot \vec{x}_2 = 0$ ✓

c. Find P and D that orthogonally diagonalize A. Note that since P is an orthogonal matrix $P^{-1} = P^T$.

$$P = \begin{bmatrix} 0.4793 & 0.5708 & -0.6667 \\ 0.0409 & 0.7442 & 0.6667 \\ 0.8767 & -0.3468 & 0.3333 \end{bmatrix}, P^{-1} = \begin{bmatrix} 0.4793 & 0.0409 & 0.8767 \\ 0.5708 & 0.7442 & -0.3468 \\ -0.6667 & 0.6667 & 0.3333 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 10 \end{bmatrix}$$

Check: $A = PDP^{-1}$ ✓