AMS10 HW8 - Solutions

Problem 1 Let

$$u = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \ v = \begin{bmatrix} -3 \\ 2 \\ 2 \end{bmatrix}.$$

Compute the following quantities.

a. $u^T v$

$$[1\,3\,-2]\begin{bmatrix} -3 \\ 2 \\ 2 \end{bmatrix} = 1 \cdot (-3) + 3 \cdot (2) - 2 \cdot (2) = -1$$

b. $v^T u$

$$v^T u = u^T v = -1$$

c.. $\left(\frac{u^T u}{v^T u}\right) u$

$$[1\,3\,-2]\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} = 1 \cdot (1) + 3 \cdot (9) - 2 \cdot (-2) = 14$$

$$\left(\frac{u^T u}{v^T u}\right) u = \left(\frac{14}{-1}\right) u = -14 \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -14 \\ -42 \\ 28 \end{bmatrix}$$

d. $||u - v||$

$$\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} - \begin{bmatrix} -3 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ -4 \end{bmatrix}$$

$$||u - v|| = \sqrt{2 + 1^1 + (-4)^2} = \sqrt{33}$$

Problem 2 Determine if the following pair of vectors are orthogonal.

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2.5 \end{bmatrix} \right\},$$

$$[2\,1\,-2]\begin{bmatrix} 2 \\ 1 \\ 2.5 \end{bmatrix} = 2 \cdot (2) + 1 \cdot (1) - 2 \cdot (2.5) = 0 \text{ orthogonal}$$

$$\left\{ \begin{bmatrix} -3 \\ 7 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -8 \\ 25 \\ -7 \end{bmatrix} \right\},$$

$$[-3\,7\,4\,0]\begin{bmatrix} 1 \\ -8 \\ 25 \\ -7 \end{bmatrix} = -3 \cdot (1) + 7 \cdot (-8) + 4 \cdot (25) + 0 \cdot -7 = -41 \text{ not orthogonal}$$

$$\left\{ \begin{bmatrix} 13 \\ -3 \\ -7 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$[13\ -3\ -7\,4] \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 13 \cdot (0) - 3 \cdot (0) - 7 \cdot (0) + 4 \cdot 0 = 0 \text{ orthogonal}$$

Problem 3 Let $W = colsp(A)$, where

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 1 \end{bmatrix}.$$

Find $W^{\perp}$, the orthogonal complement of W. We begin by taking the transpose of A and then finding the null space.

$$A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \end{bmatrix}$$

$$A^T x = 0 \implies \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 = -2x_2$$
$$x_3 = 0.$$

$$x = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$$W^{\perp} = span\{x\}$$

Problem 4 Let

$$u = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix},$$

and $W = span\{u\}$. What is the dimension of $W^{\perp}$, the orthogonal complement of W. We use the fact that

$$W + W^{\perp} = 3 \implies W^{\perp} = 3 - 1$$

since $dim(W) = 1$.

Problem 5 Let A be a 7×5 matrix. What is the smallest possible dimension of $[colsp(A)]^{\perp}$?

We use the fact that

$$W + W^{\perp} = 7$$

given $W = dim(colsp(A))$, since the column vectors are in $\mathbb{R}^7$. If A is a 7×5 matrix, then the maximum number of linearly independent column vectors will be 5, this means the dimension of the column space will be at most 5 as well and so $dim([colsp(A)]^{\perp}) \geq 2)$.

Problem 6 Let

$$u_1 = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}, \; u_2 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \; u_3 = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, \text{ and } x = \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix}.$$

Express x as a linear combination of $\{u_1, u_2, u_3\}$. For this problem we compute the coefficients such that

$$c_1 \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix}.$$

since the vectors are orthogonal it follows that

$$c_i = \frac{u_i^T x}{u_i^T u_i}$$

$$c_1 = \frac{u_1 \cdot x}{u_1 \cdot u_1} = \frac{24}{18} \tag{148}$$

$$c_2 = \frac{u_2 \cdot x}{u_2 \cdot u_2} = \frac{2}{9} \tag{149}$$

$$c_3 = \frac{u_3 \cdot x}{u_3 \cdot u_3} = \frac{10}{18} \tag{150}$$

Problem 7 Let

$$W = span \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\} \text{ and } \vec{y} = \begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix}$$

Find $\text{proj}_W \vec{y}$.

We use the fact that the vectors are orthogonal. Let

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \text{ and } u_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

then

$$\begin{aligned} \text{proj}_W \vec{y} &= \frac{u_1 \cdot \vec{y}}{u_1 \cdot u_1} u_1 + \frac{u_2 \cdot \vec{y}}{u_2 \cdot u_2} u_2 \\ &= \frac{3}{6} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + \frac{-6}{3} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1.5 \\ -1.5 \\ 3 \end{bmatrix} \end{aligned}$$