2 Week 1: Tuesday 1/10/17

2.1 Example 1

I am going to propose you guys are ready to solve a problem with only Monday’s lecture. As a warm up we will break into groups and solve the following problem: Find the steady state profile of the following PDE

\[ U_t = k T_{xx} \quad (24) \]

\[ U(x, 0) = f(x) \quad (25) \]

\[ U(0, t) = T_1 \quad (26) \]

\[ U(L, t) = T_2 \quad (27) \]

Hint: steady state implies no changes in time. In steady-state, we have \( U_t = 0 \), so we get \( U_{xx} = 0 \).

\[ U(x, \infty) = T_1 + (T_2 - T_1) \frac{x}{L} \]

Stay in groups. One more exercise lead by me in the classroom. Parts to be discussed in groups. Have one student from each group come up with a proposed answer.

\[ U_t = k U_{xx} \quad (28) \]

\[ U(x, 0) = \sin \left( \frac{\pi x}{L} \right) \quad (29) \]

\[ U(0, t) = 0 \quad (30) \]

\[ U(L, t) = 0 \quad (31) \]

Recall from separation of variable we get

\[ T' = \lambda k T \quad (32) \]

\[ X'' = \lambda x \quad (33) \]

So it is straightforward to see

\[ T' = \lambda k T \implies T(t) = C e^{\lambda kt} \]

Now we solve

\[ X'' = \lambda x \quad \text{for} \ 0 \leq x \leq L \]

We need to consider the cases \( \lambda < 0, \lambda = 0, \lambda > 0 \). One can show that \( \lambda > 0 \) and \( \lambda = 0 \) only result in a trivial solution. For \( \lambda > 0 \) we get the general solution \( x(t) = A e^{\sqrt{\lambda} x} + B e^{-\sqrt{\lambda} x} \). Applying the boundary conditions, the homogeneous linear system

\[ x(0) = A + B = 0 \quad (34) \]

\[ x(L) = A e^{\sqrt{\lambda} L} + B e^{-\sqrt{\lambda} L} \quad (35) \]

has a non-trivial solution if the determinant

\[
\begin{vmatrix}
1 & 1 \\
e^{\sqrt{\lambda} L} & e^{-\sqrt{\lambda} L}
\end{vmatrix}
\]
is zero, which we find to only hold true for $\lambda = 0$. For $\lambda = 0$ it must be that $X'' = 0 \implies X = 0$.

So we check conditions for $\lambda < 0$, which gives two complex eigenvalues and the general solution $X(x) = A \cos(\sqrt{\alpha}x) + B \sin(\sqrt{\alpha}x)$, where $\lambda = -\alpha$. Applying the boundary conditions we get

\begin{align*}
X(0) &= A = 0 \\
X(L) &= B \sin(\sqrt{\alpha}L) = 0
\end{align*}

The $\alpha$’s that make the second equation vanish must satisfy $\sqrt{\alpha} = n\pi = \frac{n\pi}{L}$ for $n = 1, 2, \ldots$.

\[X_n(x) = \sin\left(\frac{n\pi x}{L}\right), T_n(t) = e^{-\left(\frac{n\pi}{L}\right)^2 kt}\]

Therefore the general solution is some linear combination of the set

\[U_n(x, t) = \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 kt}\]

The last constraint, the initial condition will provide a unique solution.

### 2.2 generalized Fourier series expansion

#### 2.2.1 orthogonal functions

A set of orthogonal functions $\phi_n(x)$ is termed complete in the closed interval $x \in [a, b]$ if, for every piecewise continuous function $f(x)$ in the interval, the minimum square error

\[E_n \equiv ||f - (c_1 \phi_1 + \cdots + c_n \phi_n)||^2\]

(where $||f||$ denotes the L2-norm with respect to a weighting function $w(x)$) converges to zero as $n$ becomes infinite. Symbolically, a set of functions is complete if

\[
\lim_{m \to \infty} \int_a^b \left( f(x) - \sum_{n=0}^{m} a_n \phi_n(x) \right)^2 w(x) dx = 0,
\]

where the above integral is a Lebesgue integral.

#### 2.2.2 special property

\[
\int_R \phi_m(x) \phi_n(x) w(x) dx = c_m \delta_{mn}
\]

over a range $R$, where $w(x)$ is a weighting function, $c_m$ are given constants and $\delta_{mn}$ is the Kronecker delta.

We can then express a function $f(x)$ in the interval $x \in [a, b]$ by

\[f(x) = \sum_{m=0}^{\infty} a_m \phi_m(x)\]

we plug this into the orthogonality relationship (show work)

\[
\int_R f(x) \phi_n(x) w(x) dx = a_n c_n
\]
if a series for $f(x)$ of the assumed form exists, its coefficients will satisfy

$$a_n = \frac{1}{c_n} \int_R f(x) \phi_n(x) w(x) dx$$

Using a complete biorthogonal system, we can express a function $f(x)$ as follows

$$f(x) = e + \sum_{n=1}^{\infty} a_n f_1(n, x) + \sum_{n=1}^{\infty} b_n f_2(n, x)$$

A complete biorthogonal system satisfies

$$\int_R f_1(m, x) f_1(n, x) w(x) dx = c_m \delta_{mn}$$

$$\int_R f_2(m, x) f_2(n, x) w(x) dx = d_m \delta_{mn}$$

$$\int_R f_1(m, x) f_2(n, x) w(x) dx = 0$$

$$\int_R f_1(m, x) w(x) dx = 0$$

$$\int_R f_2(m, x) w(x) dx = 0$$

The coefficients are given by

$$a_n = \frac{1}{c_n} \int_R f(x) f_1(n, x) w(x) dx$$

$$b_n = \frac{1}{d_n} \int_R f(x) f_2(n, x) w(x) dx$$

$$e = \frac{\int_R f(x) w(x) dx}{\int_R w(x) dx}$$

### 2.2.3 Fourier series

The functions

$$f_1(n, x) = \cos(nx)$$

$$f_2(n, x) = \sin(nx)$$

form a complete orthogonal system over $[-\pi, \pi]$ with weighting function $w(x) = 1$ and $c_m = d_m = \pi$

Let’s revisit the example problem we did for general $f(x)$ and assume problem can be expressed as a sum of sines (learn how to do this later). Then substitute $x = \pi \hat{x}/L$ to find coefficients for solution.

### 2.3 Sturm-Liouville Eigenvalue problems

In this course these problems will arise from separation of variables. This provides a method of finding solutions to a subset of the problems.
Consider the real second-order linear differential equation of the form

\[
\frac{d}{dx} \left[ p(x) \frac{d\phi}{dx} \right] + q(x)\phi = -\lambda w(x) \phi
\]

where \( \phi \) is a function of the free variable \( x \). \( p, q, \) and \( w \) are continuous in \( a \leq x \leq b \). \( p \) is continuously differentiable in \( a < x < b \) and \( p > 0, w > 0 \) in \( a \leq x \leq b \).

boundary conditions

\[
\beta_1 \phi(a) + \beta_2 \frac{d\phi}{dx}(a) = 0 \quad (49)
\]
\[
\beta_3 \phi(b) + \beta_4 \frac{d\phi}{dx}(b) = 0 \quad (50)
\]

\( \beta_1, \ldots, \beta_4 \) are real numbers

The Sturm-Liouville (S-L) problem involves finding the values \( \lambda \) (eigenvalues) for which there exists a non-trivial solution satisfying the boundary conditions. The corresponding solution are the eigenfunctions of the problem. This is not a PDE but appears in PDEs are separable. Under the stated assumptions we have

- There are an infinite set of real eigenvalues \( \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots, \lambda_n \to \infty \) as \( n \to \infty \)
- The normalized eigenfunctions form a complete orthogonal system.

Note that the heat equation is an S-L problem with \( p(x) \equiv \text{const.} \) and \( q(x) = 0 \).

### 2.4 Example 2

Suppose at \( t = 0 \), the end at \( x = 0 \) of a bar at uniform temperature \( T_0 \) is raised to \( T_1 \) and kept at that value. The other end is insulates. Describe the temperature evolution. The system is describe by the following equations:

\[
T_t = kT_{xx} \quad (51)
\]
\[
T(x, 0) = T_0 \quad (52)
\]
\[
T(0, T) = T_1 \quad (53)
\]
\[
T_x(1, t) = 0. \quad (54)
\]

Consider \( w(x, t) = T(x, t) - T_1(x) \). Then we have

\[
w_t = kw_{xx} \quad (55)
\]
\[
w(0, t) = 0 \quad (56)
\]
\[
w_x(1, t) = 0. \quad (57)
\]

Proposing \( w(x, t) = f(x)g(t) \) and applying separation of variables, we get

\[
f''(x) + \lambda f(x) = 0 \quad (58)
\]
\[
g'(t) + \lambda kg(t) = 0 \quad (59)
\]

We have already shown that the general solution for \( f(x) \) is

\[
f(x) = A \sin(\lambda^{1/2}x) + B \cos(\lambda^{1/2}x)
\]
Applying the left boundary condition \( \implies B = 0 \). Applying the right boundary condition \( f'(1) = 0 \implies f'(1) = \lambda^{1/2} \cos(\lambda^{1/2}) = 0 \implies \lambda = (2n - 1)^2 \left( \frac{\pi}{2} \right)^2 \). This has the corresponding eigenfunctions

\[
\begin{align*}
g_n(t) &= e^{-k(n-1/2)^2 \pi^2 t} \\
f_n(x) &= \sin((n - 1/2)\pi x)
\end{align*}
\]

which must form a complete orthogonal system so the general solution is given by

\[
w(x, t) = \sum_{n=1}^{\infty} a_n \sin((n - 1/2)\pi x)e^{-k(n-1/2)^2 \pi^2 t}.
\]

We apply the initial condition and know that there are coefficients \( a_n \) s.t. \( w(x, 0) = T_0 - T_1 = \sum_{n=1}^{\infty} a_n \sin((n - 1/2)\pi x) \). We apply the special property to find the coefficients by multiplying the left and right side by \( \sin((m - 1/2)\pi x) \) and integrating over the length of the rod from \( x = 0 \) to 1

\[
\int_0^1 (T_0 - T_1) \sin((m - 1/2)\pi x) dx = a_m \int_0^1 \sin^2((m - 1/2)\pi x) dx \implies a_m = \frac{2(T_0 - T_1)}{\pi(n - 1/2)}
\]

\[
\implies T = T_1 + \frac{2}{\pi} (T_0 - T_1) \sum_{n=1}^{\infty} \frac{\sin((n - 1/2)\pi x)e^{-k(n-1/2)^2 \pi^2 t}}{(n - 1/2)}
\]