3 Week 1: Friday 1/12/17

3.1 Fourier series continued

In order for us to be able to represent \( f(x) \) in a Fourier series expansion, the function \( f(x) \) must be piecewise smooth.

**Definition. Piecewise smooth.** For \( f \) define on interval \([a,b]\), \( f \) is piecewise smooth on \([a,b]\) if there is a partition on \([a,b]\), \( \{x_j\}_{j=0}^{p}, a = x_0 < x_1 < \cdots < x_p = b \), such that \( f \) is continuously differential on each subinterval \((x_j, x_{j+1})\), and at each \( x_j \), \( f \) or its derivative \( f' \) has at most a jump discontinuity.

Draw some figures to illustrate.

**Theorem.** If \( f(x) \) is piecewise smooth on the interval \(-L \leq x \leq L\), then the Fourier series of \( f(x) \) converges

1. to the periodic extension of \( f(x) \), where the periodic extension is continuous;

2. to the average of the two limits, usually

\[
\frac{1}{2} [f(x^+) + f(x^-)],
\]

where the periodic extension has a jump discontinuity

Recall that in the heat equation we always get a Fourier sine series expansion. Up until now we have assumed that the initial condition can also be expressed as a sine series. So when we apply our initial condition we have

\[
U(x, 0) = \sum_{n=1}^{\infty} a_n \sin \left( \frac{n\pi x}{L} \right) = f(x)
\]

and we solve for the coefficients using the special property of orthogonal functions.

We have \( f(x) \) defined on \([0, L]\). If we extend it to the left in such a way that the function becomes odd \( \hat{f}(-x) = -\hat{f}(x) \) then when we solve for the coefficients

\[
a_n = \frac{1}{L} \int_{-L}^{L} \hat{f}(x) \sin \frac{n\pi x}{L} dx
\]

we know that the integrand is even (odd \( \times \) odd = even) and so this is equal to

\[
\Rightarrow a_n = \frac{2}{L} \int_{0}^{L} \hat{f}(x) \sin \frac{n\pi x}{L} dx = \frac{2}{L} \int_{0}^{L} f(x) \sin \frac{n\pi x}{L} dx.
\]

Although less common, if a Fourier cosine series is at hand, then a similar argument can be applied (even \( \times \) even = even). Cosine is an even function so we simply consider an "odd" extension of \( f(x) \) to the left and the same criteria applies. We then have

\[
b_n = \frac{2}{L} \int_{0}^{L} f(x) \cos \frac{n\pi x}{L} dx
\]

So we are starting with a function on domain \([0,L]\). If our function is a pure sine series then we consider an odd extension to the left.....if even (cosine) then we consider an even extension to the left. This function is repeated in intervals relating to a period of 2L
3.2 Method of eigenfunction expansion for non-homogeneous equations

Note: We are assuming continuous $f(x)$. Now that we know we can extend any continuous function $f(x)$ in our region of interest $[0, L]$ such that it can be represented by a Fourier sine series, or Fourier cosine series we can apply the method of eigenfunction expansion to non-homogeneous equations.

We expand the unknown solution $U(x, t)$ in terms of the eigenfunctions of the homogeneous problem. One of the eigenfunctions that continues to appear through the heat equation is $f_n(x) = \sin \left(\frac{n\pi x}{L}\right)$. We can no longer assume $g_n(t) = e^{-\left(\frac{n\pi}{L}\right)^2 kt}$. We assume that the solution will have the form

$$U(x, t) = \sum_{n=1}^{\infty} a_n(t) \sin \frac{n\pi x}{L}.$$  

We can check whether a solution of this form can satisfy the PDE by plugging it back into the heat equation.

Consider

$$U_t = kU_{xx} + s(x, t), \quad 0 \leq x \leq L$$  

$$U(x, 0) = f(x)$$  

$$U(0, t) = 0$$  

$$U(L, t) = 0, \quad 0 \leq x \leq L$$

$$\sum_{n=1}^{\infty} a_n'(t) \sin \frac{n\pi x}{L} = -k \sum_{n=1}^{\infty} a_n(t) \left(\frac{n\pi}{L}\right)^2 \sin \frac{n\pi x}{L} + s(x, t)$$

Now we know we can express $s(x, t)$ as a sine series:

$$s(x, t) = \sum_{n=1}^{\infty} S_n(t) \sin \left(\frac{n\pi x}{L}\right)$$

which will converge to the function for $0 \leq x \leq L$.

We now move the terms to the left-hand side

$$\sum_{n=1}^{\infty} \left[a_n'(0) + k a_n(t) \left(\frac{n\pi}{L}\right)^2 - S_n(t)\right] \sin \frac{n\pi x}{L} = 0$$

So then we find $a_n(t)$ by solving the system

$$a'_n + \frac{n^2 \pi^2}{L^2} k a_n = S_n(t)$$

From the initial condition we have

$$U(x, 0) = \sum_{n=1}^{\infty} a_n(t) \sin \frac{n\pi x}{L} = f(x)$$

Again, we expand $f(x)$ in a sine series

$$f(x) = \sum_{n=1}^{\infty} f_n \sin \frac{n\pi x}{L} \implies a_n(0) = f_n.$$
We can solve this ODE (IVP).
Recall for system of the form
\[ x'(t) = ax(t) + bu(t) \]
where \( u \) is the input, \( x \) is the dependent variable, and \( a \) and \( c \) are constant coefficients, given initial condition \( x(0) = x_0 \) and arbitrary input \( u(t) \) defined on \( [0, \infty) \), the solution is
\[ x(t) = e^{at}x_0 + \int_0^T e^{a(t-\tau)}bu(\tau)d\tau \]

### 3.3 Complex Fourier series

A more compact way to write the Fourier series using complex functions
\[ f(x) = \sum_{n=-\infty}^{\infty} c_n e^{-in\pi x/L} \]
this is a complete orthogonal system and has the special property
\[ \int_{-L}^{L} e^{-im\pi x/L}e^{-in\pi x/L}dx = 2L \delta_{mn}, \]
where the overline represents the complex conjugate. This leads to the coefficients
\[ c_n = \frac{1}{2L} \int_{-L}^{L} f(x)e^{-im\pi x/L}dx \]
if \( f(x) \) is real, then \( c_{-n} = \overline{c_n} \)
or equivalently
\[ c_n = \frac{1}{2\pi} \int_0^{2\pi} f(\theta)e^{-im\theta}d\theta \]

Solve the Poisson equation
\[ \nabla^2 U = g(r, \theta) \]
\[ U(a, \theta) = f(\theta) \] (67)
where
\[ \nabla^2 U = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial U}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 U}{\partial \theta^2} \]
in polar coordinates
We are going to apply the principle of superposition by solving two different problems
\[ \nabla^2 U_1 = 0 \]
\[ U_1(a, \theta) = f(\theta) \] (69)
and
\[ \nabla^2 U_2 = g(r, \theta) \]
\[ U_2(a, \theta) = 0 \] (71)
So our final solution will be the sum of the two $U = U_1 + U_2$, which holds by the principle of superposition.

Applying separation of variables $U = R(r)H(\theta)$

$$H'' + \lambda H = 0$$

we know the eigenfunctions $H$: $\cos(n\theta), \sin(n\theta)$ where we now prescribe periodic boundary conditions based on the physics of the problem

$$H(-\pi) = H(\pi) \quad (72)$$
$$H'(-\pi) = H'(\pi) \quad (73)$$

Note there is no external forcing or input so this system has homogeneous boundary conditions so the eigenfunctions will form a complete set. We already know from previous examples that the eigenfunctions of the homogenous problem will consist of the Fourier series. So we consider the complex Fourier series expansion. Note that the boundary conditions keep both sines and cosines. The general solution and its partial derivative are

$$U(r, \theta) = \sum_{n=-\infty}^{\infty} c_n(r)e^{in\theta} \quad (74)$$

$$U_r = \sum_{n=-\infty}^{\infty} c'_n(r)e^{in\theta} \quad (75)$$

$$U_{rr} = \sum_{n=-\infty}^{\infty} c''_n(r)e^{in\theta} \quad (76)$$

$$U_{\theta\theta} = \sum_{n=-\infty}^{\infty} c_n(r)(-n^2)e^{in\theta} \quad (77)$$

If we are solving $U_1$, we apply the boundary condition to find $c_n$

$$U(a, \theta) = \sum_{n=-\infty}^{\infty} c_n(a)e^{in\theta} = \sum_{n=-\infty}^{\infty} f_ne^{in\theta}$$

where we assume such coefficients $f_n$ exist then

$$f_n = \int_0^{2\pi} f(\theta)e^{-in\theta} d\theta$$

So we have $c_n(a) = f_n$ and $|c_n(0)| < \infty$.

We check to see if this solution which satisfies the initial condition can also satisfy the second problem. This is a complete system so if we expand the forcing function

$$g(r, \theta) = \sum_{n=-\infty}^{\infty} g_n(r)e^{in\theta}$$
then plugging this back into the PDE and grouping terms we get

\[
U_{rr} + \frac{1}{r} U_r + \frac{1}{r^2} U_{\theta \theta} = g(r, \theta) \tag{78}
\]

\[
r^2 U_{rr} + r U_r + U_{\theta \theta} = g(r, \theta) r^2 \tag{79}
\]

\[
\sum_{n=-\infty}^{\infty} \left( r^2 c''_n(t) + r c'_n(t) - n^2 c_n \right) e^{in\theta} = r^2 \sum_{n=-\infty}^{\infty} g_n(r) e^{in\theta} \tag{80}
\]

\[
\implies r^2 c''_n + r c'_n - n^2 c_n = g_n(r) r^2 \tag{81}
\]

Example with

\[
g(r, \theta) = \sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}
\]

\[
g_n(r) = \begin{cases} 
\pm \frac{1}{2i}, & \text{for } n = \pm 1 \\
0, & \text{otherwise}
\end{cases}
\]

Solve ODEs: \( n \neq \pm 1 \)

\[
r^2 c''_n + r c'_n - n^2 c_n = 0
\]

\[
c_n(0) = \text{bounded} \\
c_n(a) = 0
\]

\[
\implies c_n = 0 \tag{82}
\]

For \( n = \pm 1 \)

\[
r^2 c''_n + r c'_n - c_n = \pm \frac{r^2}{2i}
\]

general solution to the homogeneous equation is \( k_1 r \). Now we need to find a particular solution to the non-homogeneous equation. Try \( c_n(r) = A r^2 \)

\[
r^2 A + 2r^2 A - r^2 A = 3A r^2 = \pm \frac{r^2}{2i} \implies A = \pm \frac{1}{6i}
\]

then we have

\[
c_n(r) = \pm \frac{r^2}{6i} + k_1 r
\]

apply initial condition

\[
c_n(a) = \pm \frac{a^2}{6i} + k_1 a \implies k_1 = \mp \frac{a}{6i}
\]

this gives

\[
c_n(r) = \pm \frac{1}{6i} \left( r^2 - ar \right)
\]

Final solution

\[
U(r, \theta) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta} = \frac{e^{i\theta}}{6i} \left( r^2 - ar \right) - \frac{e^{-i\theta}}{6i} \left( r^2 - ar \right)
\]

### 3.4 Reaction-diffusion system (Supplemental)

Reaction diffusion system given by

\[
u_t = D \nabla^2 u + R(u)
\]

For example for the FitzHugh-Nagumo equation (activator-inhibitor system)

\[
u_t = d_u \nabla^2 u + f(u) - \sigma v \tag{83}
\]

\[
\tau v_t = d_v \nabla^2 v + u - v \tag{84}
\]
where \( f(u) = \lambda u - u^3 - \kappa \).

For ease of analysis assume a system of one dimensional space. We linearize the system
\[
\frac{\partial \tilde{u}}{\partial t} = D \frac{\partial^2 \tilde{u}}{\partial x^2} + A \tilde{u}
\]
where
\[
A_{ij} = \left. \frac{\partial R_i}{\partial u_j} \right|_{u = u^*}
\]
where \( u^* \) denotes the equilibrium point about which we are linearizing.

We derive conditions for Turing instability. Consider the linearized general PDE system
\[
\frac{\partial \tilde{u}}{\partial t} = D \frac{\partial^2 \tilde{u}}{\partial x^2} + A \tilde{u} \tag{85}
\]
where \( A \in \mathbb{R}^{(n \times n)} \), \( \tilde{u} \in \mathbb{R}^n \), and \( D \in \mathbb{R}^{(n \times n)} \) is a diagonal matrix. We apply a method of separation of variables. We assume the solution can be written as
\[
\tilde{u} = \phi(x) \bar{g}(t),
\]
where \( \bar{g}(t) \) is a diagonal matrix with entries of \( \tilde{g}(t) \) along the diagonal and \( \bar{g} = [1, \ldots, 1]^T \). Substituting \( \tilde{u} \) into equation \((85)\) gives
\[
\phi(x) \frac{\partial \bar{g}}{\partial t} = \frac{\partial^2 \phi(x)}{\partial x^2} \bar{g} + A \phi(x) \bar{g}
\]
\[
= G \frac{\partial^2 \phi(x)}{\partial x^2} \bar{g} + A \phi(x) \bar{g} \tag{87}
\]
where the last equality holds because \( D \) and \( G \) commute. Rearranging terms we get
\[
G^{-1} \left( \frac{\partial G}{\partial t} - AG \right) \bar{g} = -D^{-1} \frac{\partial^2 \phi(x)}{\partial x^2} \bar{g} = [\lambda_1, \lambda_2, \ldots, \lambda_n]^T \tag{88}
\]
We assume \( G \) is an invertible matrix. This implies \( \bar{g} > 0 \). Assuming the time-varying function is exponential, then we will have that \( \bar{q} \geq 0 \) and \( \bar{q} = 0 \) only in the limit as \( t \to \infty \). We can solve for the eigenvalues by solving for \( \phi(x) \). With the eigenvalues known, we can then solve for the time-varying functions.

We denote \( \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \) by \( \Lambda \). Therefore, for Turing instability we look for unstable eigenvalues of the matrix \( A + \Lambda \). Assuming Neumann boundary conditions
\[
\frac{\partial u}{\partial x}(0, t) = 0 \tag{94}
\]
\[
\frac{\partial u}{\partial x}(L, t) = 0, \tag{95}
\]
for $\mathbf{D} = \text{diag}(d_1, d_2, \ldots, d_n)$, the corresponding eigenvalues are $\lambda_i(k) = -d_i \left( \frac{k \pi}{L} \right)^2$ with eigenfunctions $\phi(x) = \cos \left( \frac{k \pi x}{L} \right)$. Just as presumed, the different scaling terms in $\mathbf{D}$ do not change the eigenfunction. The linear scaling appears in the eigenvalues. The general solution is given by

$$\tilde{u} = \sum_{k=0}^{\infty} A_n \cos \left( \frac{k \pi x}{L} \right) \bar{g}(t, k).$$

(96)

Therefore, $\phi(x)$ is bounded and so we look for instability of $\bar{g}(t, k)$ for the different spacial modes. Note that the non-zero terms in $\Lambda$ monotonically increase with the variable $k$. For pattern formation, we do not want the eigenvalues to approach values in the right half plane as $k \to \infty$. We require that there are a finite number of unstable modes. For analysis we can consider a controls approach to analyzing the transfer function $(s \mathbf{I} - \mathbf{A} - \Lambda)^{-1}$ or equivalently looking at the system $(s \mathbf{I} - \mathbf{A})^{-1}$ with the feedback gain $\Lambda$. Criteria such as Nyquist criterion can help relate right half plane poles and zeros to stability of the system.