Midterm Review

1 Complex numbers

$z = a + ib \Leftrightarrow z = re^{i\theta} \Leftrightarrow z = r(\cos\theta + i\sin\theta)$

$z^{1/m}$? $\quad z^m$

$z = \sqrt{2}\, e^{\frac{\pi}{3} i} = \sqrt{2}\, e^{(\frac{\pi}{3} + 2\pi k) i}$

z^{11} ? $\Rightarrow z^{11} = 2^{11/2} e^{(\frac{\pi}{3} + 2\pi k) i \cdot 11}$

$= 2^{11/2} e^{(11\pi/3 + 22\pi k) i}$

$= 2^{11/2} e^{11\pi/3\, i}$

$z^{1/11} = 2^{\frac{1}{22}} e^{(\pi/3 + 2\pi k) i / 11}$

$= 2^{1/22} e^{(\frac{\pi}{33} + \frac{2}{11}\pi k) i}$

$\Rightarrow$ 11 distinct roots $\quad k = 0, 1, 2, \dots, 10$

$z = a + ib$

$a = \operatorname{Re}(z^{1/11}) = 2^{1/22} \cos\left(\frac{\pi}{33} + \frac{2}{11}\pi k\right)$

$b = \operatorname{Im}(z^{1/11}) = 2^{1/22} \sin\left(\frac{\pi}{33} + \frac{2}{11}\pi k\right)$

2 Know how to do

matrix/vector products

matrix/matrix products

Suppose $A \in \mathbb{R}^{3\times5}$, $B \in \mathbb{R}^{4\times3}$

Is AB well defined?

$(3\times5)(4\times3) = \times$!

$BA \quad (4\times3)(3\times5) = (4\times5)$

3 ID echelon form / pivot variables / free variables

4 Row equivalence / Gussian elimination

- know all 3 elementary operations
- note rowspace is invariant under elementary operations (row equivalence) (NOT true for the column vectors/space)

5 Matrix inverse
Be able to compute A^{-1} for 2x2, maybe 3x3?
2x2 ⇒ use formula
3x3 ⇒ use Gussian elimination method

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ ~~det(A)~~ $A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

- know when the inverse exists!
  - A must be a square matrix
  - A must be full rank
    if $A \in \mathbb{R}^{n \times n} \Rightarrow \text{rank}(A) = n$
    all column/row vectors are linearly independent
  - $\det(A) \neq 0$

- What is A^{-1} good for? Solving linear system $Ax = b$

$A^{-1}Ax = A^{-1}b$

$Ix = A^{-1}b \Rightarrow \underline{x} = A^{-1}b$

A^{-1} for $A \in \mathbb{R}^{3\times 3}$

$$\underbrace{\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}}_{A} \left| \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \right.$$

Apply Gussian elimination on the left-hand side (similar to when we solved $Ax=b$ with augmented matrix $[A\ b]$)

Keep applying operations until the left-side looks like $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

6 Existence and uniqueness of solutions to $Ax=b$

Step 1 Create augmented matrix $[A\ b]$

Step 2 Reduce to echelon form

Step 3 Look for consistency (if inconsistent $\Rightarrow$ no solution)

Step 4 Look for free variables

Step 5 ≥ 1 free variables $\Rightarrow$ infinite soltns

0 " " $\Rightarrow$ unique soltn

- know how to find the solution set of $A\vec{x} = \vec{b}$

Once you have reduced to echelon form, solve for unknowns as a function of free variables if there are any inconsistent

Example

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \qquad b = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

$$[A\, b] = \left[\begin{array}{cc:c} 1 & 2 & 5 \\ 0 & 0 & 2 \end{array}\right] \Rightarrow \begin{array}{l} x_1 + 2x_2 = 5 \\ \underline{\underline{0 = 2}} \end{array}$$

inconsistent

$\Rightarrow$ no solution

$A \in \mathbb{R}^{5\times 3} \qquad Ax = b \qquad x \in \mathbb{R}^{3\times 1}$

reduced row echelon form is

$$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

7 Linear dependence / independence

vectors $\{v_1, v_2, \ldots v_n\}$ are linearly dependent if $\exists$ $k_1, k_2, \ldots, k_n$ <u>not all</u> 0 s.t. $k_1 v_1 + k_2 v_2 + \ldots + k_n v_n = 0$

Special cases

2D vectors – we can apply a graphical representation

Consider a set of <u>two vectors</u>

If the lie on the same line $\Rightarrow$ dependent

otherwise $\Rightarrow$ independent

2 vectors in $\mathbb{R}^2$ that are linearly independent span $\mathbb{R}^2$

3 vectors in $\mathbb{R}^2$ are <u>always</u> linearly dependent

In general, n vectors in $\mathbb{R}^n$ that are linearly indep. span $\mathbb{R}^n$

$\Rightarrow$ $n+1$ vectors are <u>always</u> linearly dependent

If a set contains $\vec{0}$, then it is linearly dependent

- know the transpose of a matrix

$$A = \begin{bmatrix} 1 & 5 \\ 0 & 4 \\ 2 & 3 \end{bmatrix} \quad A^T = \begin{bmatrix} 1 & 0 & 2 \\ 5 & 4 & 3 \end{bmatrix}$$

8 Spans / basis / subspace

Suppose matrix $A \in \mathbb{R}^{11 \times 9}$

Is it possible that the matrix when reduced to echelon form has a pivot point in every row? No

$$\dim(\text{colsp}(A)) = \dim(\text{rowsp}(A)) \leq 9$$

≥ 2 rows of zero when reduced

maximum 9 pivot points

Max # linearly indep. rows
$\equiv$ " " " " columns
$\equiv$ rank (A)
$\equiv$ # of pivot points/variables in echelon form

Ex. Suppose $\vec{a}_1, \dots, \vec{a}_n$ are in $\mathbb{R}^m$
how ~~[illegible]~~ to check if span $\{\vec{a}_1, \dots \vec{a}_n\} = \mathbb{R}^m$

Requirements to span $\mathbb{R}^m$
need m linearly independent vectors
How do we check

Step 1 $n \geq m$

Step 2 Generate a matrix with vectors
$A = [\vec{a}_1, \dots, \vec{a}_n]$

rank(A) = rank(A^T) = dim(colsp(A)) = dim(rowsp(A))

Step 3 row reduce to echelon form

Step 4 determine rank (A)
(count up the pivot points)

Step 5 If # of pivots = m
$\Rightarrow$ span $\{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n\} = \mathbb{R}^m$

Consider

$$C = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ -1 & -2 & -3 & -4 & -5 \end{bmatrix} \Rightarrow \text{rank}(C) = 1$$

$$\text{colsp}(C) = \text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \end{bmatrix}, \dots \begin{bmatrix} 5 \\ -5 \end{bmatrix}\right\}$$

rank(C) ≤ 2

$\dim(\text{colsp}(C)) = 1$

$\Rightarrow$ basis for colsp(C) has 1 element

basis is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\text{colsp}(C) = \text{Span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\} = \begin{bmatrix} a \\ b \end{bmatrix} : a = -b$

Ex If $A \in \mathbb{R}^{6\times 8}$, what is the smallest possible dimension of Nul A (Ker(A))

$\text{rank}(A) \leq 6$

Ax

$(6\times 8)(8\times 1)$

"domain"

$$\underbrace{\dim(\mathbb{R}^8)}_{8} = \underbrace{\text{rank}(A)}_{\leq 6} + \dim(\text{Ker}(A))$$

$\Rightarrow \dim(\text{Ker}(A)) \geq 2$

9 Determinants

Find the value of k for which the following matrix is not invertible

$$A = \begin{bmatrix} 1 & k & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix}$$

not invertible $\Rightarrow$ det $= 0$

$$\det(A) = 1 \cdot \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} - k \begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix} + 1 \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}$$

$= 1(3-2) - k(-2) + 1(-1) = 1 + 2k - 1 = 0$ $\Rightarrow 2k = 0$, $k = 0$

Alternatively, we can ask ~~find~~ find k s.t. A is invertible
$\Rightarrow \det(A) \neq 0$

Ex Let $A = \begin{bmatrix} 2 & 1 & 1 \\ \alpha & 1 & \alpha \\ \alpha+2 & 2 & \alpha+1 \end{bmatrix}$

Find α (if it exists) that makes
$\operatorname{rank}(A) = 3 \Rightarrow$ ~~[illegible]~~
There are 3 pivot points in echelon form