Midterm Review

1) Complex numbers
   \( z = a + ib \iff z = re^{i\theta} \iff z = r(\cos\theta + i\sin\theta) \)

   \( z^m \iff z^m \)
   \( z = \sqrt{z} e^{\frac{\pi}{3}i} = \sqrt{z} e^{\left(\frac{\pi}{3} + 2\pi k\right)i} \)
   \( z^2 \iff z^2 = 2^{\frac{1}{2}} e^{\left(\frac{\pi}{3} + 2\pi k\right)i/11} \)
   \( z^{1/11} = 2^{\frac{1}{11}} e^{\left(\frac{\pi}{3} + 2\pi k\right)i} \)

   \( n \) distinct roots \( k = 0, 1, 2, \ldots, 10 \)

2) Know how to do
   matrix/vector products
   matrix/matrix products

Suppose \( A \in \mathbb{R}^{3 \times 5} \), \( B \in \mathbb{R}^{4 \times 3} \)

- Is \( AB \) well defined?
  \( (3 \times 5) (4 \times 3) = X \cdot X \)

- \( BA \)
  \( (4 \times 3) (3 \times 5) = (4 \times 5) \)
3. Row echelon form / pivot variables / free variables

4. Row equivalence / Gaussian elimination
   - know all 3 elementary operations
   - note rowspace is invariant under elementary operations (row equivalence)
     (not true for the column vectors/space)

5. Matrix inverse
   - be able to compute $A^{-1}$ for $2 \times 2$, maybe $3 \times 3$?
     $2 \times 2 \Rightarrow$ use formula
     $3 \times 3 \Rightarrow$ use Gaussian elimination method

   $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
   $\det(A) A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

   - know when the inverse exists!
     - $A$ must be a square matrix
     - $A$ must be full rank
     - if $A \in \mathbb{R}^{n \times n}$ $\Rightarrow$ rank $(A) = n$
     - all column/row vectors are linearly independent
     - $\det(A) \neq 0$

   - What is $A^{-1}$ good for? Solving linear system $Ax = b$
     $A^{-1}Ax = A^{-1}b$
     $Ix = A^{-1}b \Rightarrow x = A^{-1}b$
$A^{-1}$ for $A \in \mathbb{R}^{3 \times 3}$

\[
\begin{bmatrix}
a & b & c \\
d & e & f \\
g & h & i
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} = A
\]

Apply Gaussian elimination on the left-hand side (similar to when we solved $Ax = b$) with augmented matrix $[A \ b]$

Keep applying operators until the left-side looks like

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

6. Existence and uniqueness of solutions to $Ax = b$

   Step 1. Create augmented matrix $[A \ b]$
   Step 2. Reduce to echelon form
   Step 3. Look for consistency (If inconsistent $\Rightarrow$ no solution)
   Step 4. Look for free variables
   Step 5. $\geq 1$ free variables $\Rightarrow$ infinite solutions
   $0 " " \Rightarrow$ unique solution

- Know how to find the solution set of $A\hat{x} = \delta$

Once you have reduced to echelon form, solve for unknowns as a function of free variables if there are any
Example

\[ A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \quad b = \begin{bmatrix} 5 \\ 2 \end{bmatrix} \]

\[ (A|b) = \begin{bmatrix} 1 & 2 & | & 5 \\ 0 & 0 & | & 2 \end{bmatrix} \Rightarrow x_1 + 2x_2 = 5 \]

\[ 0 = 2 \]

inconsistent => no solution

\( A \in \mathbb{R}^{5 \times 3} \quad Ax = b \quad x \in \mathbb{R}^{3 \times 1} \)

Reduced row echelon form is

\[ \begin{bmatrix} 1 & 0 & | & 0 \\ 0 & 1 & | & 0 \\ 0 & 0 & | & 1 \\ 0 & 0 & | & 0 \end{bmatrix} \]

\( A^- = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \)

Section 7

Linear dependence/independence

Vectors \( \{v_1, v_2, \ldots, v_n\} \) are linearly dependent if \( \exists k_1, k_2, \ldots, k_n \) not all 0

such that

\[ k_1v_1 + k_2v_2 + \ldots + k_nv_n = 0 \]

Special cases

2D vectors - we can apply a graphical representation

Consider a set of two vectors \( \vec{a}, \vec{b} \)

If they lie on the same line \( \Rightarrow \) dependent

Otherwise \( \Rightarrow \) independent
2 vectors in \( \mathbb{R}^2 \) that are linearly independent span \( \mathbb{R}^2 \)

3 vectors in \( \mathbb{R}^2 \) are always linearly dependent

In general, \( \mathbb{R}^n \) vectors in \( \mathbb{R}^n \) that are linearly independent span \( \mathbb{R}^n \)

\( \Rightarrow \) \( n+1 \) vectors are always linearly dependent

If a set contains \( \vec{0} \), then it is linearly dependent

- Know the transpose of a matrix

\[
A = \begin{bmatrix}
1 & 5 \\
2 & 4 \\
3 & 5
\end{bmatrix}
\]

\[
A^T = \begin{bmatrix}
1 & 5 \\
2 & 4 \\
3 & 5
\end{bmatrix}
\]

B) Span, basis, subspace

Suppose matrix \( A \in \mathbb{R}^{11 \times 9} \)

Is it possible that the matrix when reduced to echelon form has a pivot in every row? No

\[
\dim(\text{col}(A)) = \dim(\text{row}(A)) \leq 9
\]

\( \Rightarrow \) 2 rows of zero when reduced

maximum 9 pivot points
Max # linearly indep. rows

=  " " " " columns

= \text{rank}(A)

= \# of pivot points/variables in echelon form

Ex. Suppose $\tilde{a}_1, \ldots, \tilde{a}_n$ are in $\mathbb{R}^m$

how to check if $\text{span}\{\tilde{a}_1, \ldots, \tilde{a}_n\} = \mathbb{R}^m$

Requirements to span $\mathbb{R}^m$

need $m$ linearly independent vectors

How do we check

Step 1 $n \geq m$

\[
\text{rank}(A) = \text{rank}(A^T)
\]

= $\dim(\text{colsp}(A))$  Step 2 generate a matrix with vectors $A = [\tilde{a}_1, \ldots, \tilde{a}_n]$

= $\dim(\text{rowsp}(A))$  Step 3 row reduce to echelon form

Step 4 determine $\text{rank}(A)$ (count up the pivot points)

Step 5 if $\# \text{ of pivots} = m$

$\Rightarrow \text{span}\{\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n\} = \mathbb{R}^m$

Consider

$C = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ -1 & -2 & -3 & -4 & -5 \end{bmatrix} \Rightarrow \text{rank}(C) = 1$

$\text{colsp}(C) = \text{span}\{[1], [2], \ldots, [5]\}$

$\text{rank}(C) \leq 2$
\[ \dim(\operatorname{colsp}(C)) = 1 \]
\[ \exists \text{ basis for } \operatorname{colsp}(C) \text{ has 1 element} \]
\[ \text{basis is } \begin{bmatrix} 1 \\ -1 \end{bmatrix} \]
\[ \operatorname{colsp}(C) = \text{Span} \{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \} = \begin{bmatrix} a \\ b \end{bmatrix}; \ a = -b \]

**Example:**
If \( A \in \mathbb{R}^{6 \times 8} \), what is the smallest possible dimension of \( \text{Null } A \) (kernel of \( A \))?

\[ \operatorname{rank}(A) \leq 6 \]

**Ax**
\[ (6 \times 8)(8 \times 1) \]

"domain"

\[
\dim(\mathbb{R}^8) = \operatorname{rank}(A) + \dim(\operatorname{Ker}(A)) \leq 6 \]

\[ \Rightarrow \dim(\operatorname{Ker}(A)) \geq 2 \]

**Determinants**

Find the value of \( k \) for which the following matrix is not invertible.

\[ A = \begin{bmatrix} 1 & k & 1 \\ 0 & 1 & 2 \\ 1 & 1 & k \end{bmatrix} \]

not invertible \( \Rightarrow \) \( \det = 0 \)

\[
\det(A) = 1 \cdot \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} - k \begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix} + 1 \begin{vmatrix} 0 & 1 \\ 1 & 3 \end{vmatrix} = 1 + 2k - 1 = 0 \]

\[ k = 0 \]
Alternatively, we can ask to find $k$ s.t. $A$ is invertible $\Rightarrow \det(A) \neq 0$

Ex. Let $A = \begin{bmatrix} 2 & 1 & 1 \\ \alpha & 1 & \alpha \\ d+2 & 2 & d+1 \end{bmatrix}$

Find $\alpha$ (if it exists) that makes $\text{rank}(A) = 3 \Rightarrow \alpha = \ldots$

There are 3 pivot points in Echelon Form