Lecture 10

Recap: Given a matrix A, we can decompose the matrix into row vectors or column vectors

$$\underbrace{\begin{bmatrix} - & \vec{r}_1 & - \\ - & \vec{r}_2 & - \\ & \vdots & \\ - & \vec{r}_m & - \end{bmatrix}}_{A} \qquad \underbrace{\begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & & | \end{bmatrix}}_{A}$$

1) The rank of a matrix A (rank(A)) is equal to the maximum # of linearly independent rows of A

2) The maximum number of linearly independent rows = the maximum number of linearly independent columns

3) The nonzero rows of a matrix in echelon form are linearly independent

$\Rightarrow$ Reducing a matrix to echelon form gives us the rank of a matrix

Theorem: Consider a system of linear equations in n unknowns with augmented matrix $M = [A, \vec{b}]$

then:

(a) The system has a solution if and only if $\text{rank}(A) = \text{rank}(M)$ (not inconsistent)

(b) The solution is unique if and only if the $\text{rank}(A) = \text{rank}(M) = n$

Application

Example unknowns x_1, x_2, x_3, x_4 $n = 4$

Augmented matrix for system $A\vec{x} = \vec{b}$

$$M = \left[\begin{array}{cccc|c} 1 & 1 & -2 & 4 & 5 \\ 2 & 2 & -3 & 1 & 3 \\ 3 & 3 & -4 & -2 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & 0 & -10 & -9 \\ 0 & 0 & 1 & -7 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\underbrace{\quad\quad\quad}_{A}$ $\underbrace{\quad}_{\vec{b}}$

$\text{rank}(A) = 2 \neq 4$ infinite

$\text{rank}(M) = 2 \neq 4$ solutions

Application Finding basis for rowspace and columnspace

Example: Let W be a subspace of $\mathbb{R}^4$ spanned by the following vectors

$u_1 = (1, 2, 1, 3)$, $u_2 = (1, 3, 3, 5)$

$u_3 = (3, 8, 7, 13)$ $u_4 = (1, 4, 6, 9)$

Find a basis for W

Step 1: Form a matrix M whose rows are the given vectors

$$M = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 1 & 3 & 3 & 5 \\ 3 & 8 & 7 & 13 \\ 1 & 4 & 6 & 9 \end{bmatrix}$$

Step 2: Row reduce M to echelon form

$$M \sim \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

row equivalent

Step 3: Output nonzero rows of the echelon matrix

$\vec{v}_1 = (1, 0, 0, 5)$, $\vec{v}_2 = (0, 1, 0, -2)$

$\vec{v}_3 = (0, 0, 1, 2)$

$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ form a basis for the subspace W

$\dim(W) = 3$

Example: Find a basis that is composed of the original vectors

Step 1: Form the matrix M whose columns are the given vectors

$$M = [\vec{u}_1^T, u_2^T, u_3^T, u_4^T] = \begin{bmatrix} 1 & 1 & 3 & 1 \\ 2 & 3 & 8 & 4 \\ 1 & 3 & 7 & 6 \\ 3 & 5 & 13 & 9 \end{bmatrix}$$

Step 2: Row reduce to echelon form

$$M \sim \begin{array}{c} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \quad\quad\quad k=3 \end{array}$$

Step 3:

For each column in the echelon matrix without a pivot, delete (cast out) the vector u_k from the list of vectors

$$\{u_1, u_2, \overset{\text{throw out}}{\cancel{u_3}}, u_4\}$$

Step 4: Output the remaining vectors (which correspond to columns with pivots)

basis: $\{u_1, u_2, u_4\}$

Example: Find the basis and dimension of the column space of a matrix [redacted]

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 & 1 \\ 2 & 5 & 5 & 6 & 4 \\ 3 & 7 & 6 & 11 & 6 \\ 1 & 5 & 10 & 3 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5 & 0 & 0 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

basis for colsp(A)

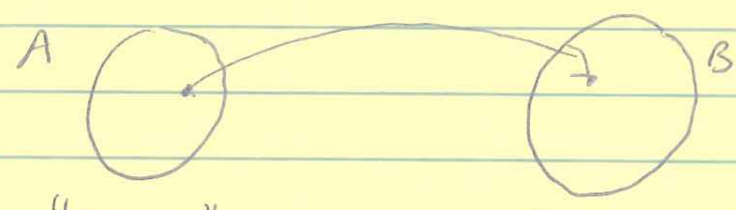
$$= \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 7 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 11 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 6 \\ 9 \end{bmatrix} \right\}$$

$\dim(\text{colsp}(A)) = 4$

Linear Mappings - Matrix Mappings

First we discuss linear mappings: Let A and B be arbitrary non-empty sets. Suppose to each element A, there is assigned a unique element in B. The collection of such assignments is called mappings (or map) from A into B, and is denoted by

$$f: A \to B$$



"domain" of mapping   "target set"

$$a \to \boxed{f} \longrightarrow b$$

$a \in A$ function/map $b \in B$

We turn our focus to Matrix mappings.
Let A be any $m \times n$ matrix over field $\mathbb{R}$, then A determines a mapping

$$F_A : \mathbb{R}^n \to \mathbb{R}^m \quad \text{by}$$

$$F_A(\vec{u}) = A\vec{u} \quad \text{where}$$

the vectors in $\mathbb{R}^m$ and $\mathbb{R}^n$ are written as columns.

$$A = \begin{bmatrix} 5 & 1 & 0 \\ 4 & -1 & 1 \\ 3 & 0 & 8 \\ 0 & 1 & 2 \end{bmatrix} \qquad F_A : \mathbb{R}^3 \to \mathbb{R}^4$$

Let $\vec{u} = \begin{bmatrix} 1 \\ 3 \\ -5 \end{bmatrix} \in \mathbb{R}^3$

$A\vec{u} \in \mathbb{R}^4$

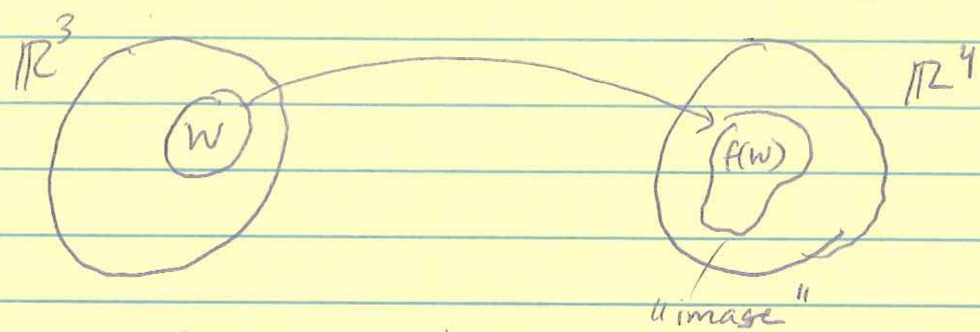
$$F_A(\vec{u}) = A\vec{u} = \begin{bmatrix} 8 \\ -4 \\ -37 \\ -7 \end{bmatrix}$$

Let W be a subspace of $\mathbb{R}^3$

$$W = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

then $f(W)$ is the image of W

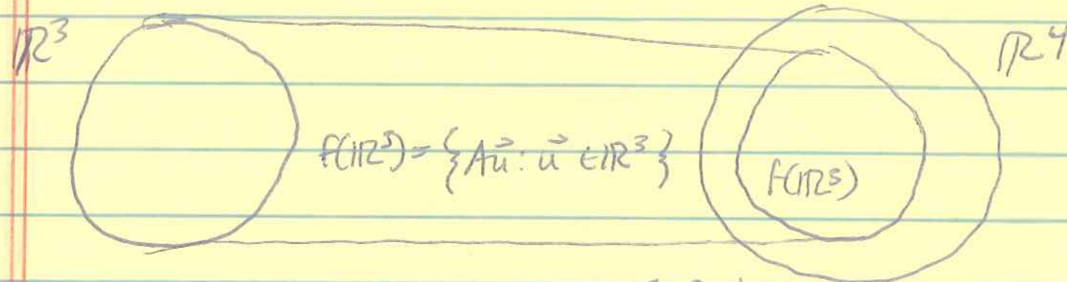
$$f(W) = \{ A\vec{u} : \vec{u} \in W \}$$



$$f(W) = \text{span}\left\{ A\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, A\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$= \text{span}\left\{ \begin{bmatrix} 5 \\ 4 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$f(\mathbb{R}^3)$ is called the image or range of f



$$f(\mathbb{R}^3) = \text{Span}\left\{ \begin{bmatrix} 5 \\ 4 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 8 \\ 2 \end{bmatrix} \right\}$$

range
Span of all column vectors

$F^{-1}(u)$ pre-image or inverse of W'

$$f^{-1}(W) = \{ (A^{-1})\vec{u} : \vec{u} \in W')$$

Let $W' = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

Note A is not a square matrix, so we can apply a modified technique to solve for x

$$\text{transpose} \rightarrow A^T (A\vec{x} = \vec{b})$$

$$\underbrace{(A^TA)}_{\text{guaranteed square matrix}}\vec{x} = A^T\vec{b}$$

$$\vec{x} = (A^TA)^{-1} A^T \vec{b}$$

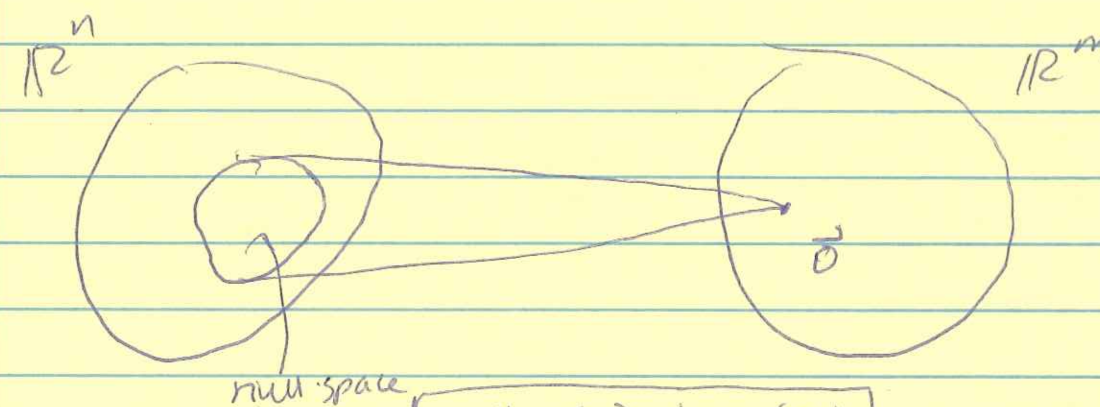
$$(A^T A)^{-1} A^T \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.1246 \\ 0.1310 \\ -0.0551 \end{bmatrix}$$

$$f^{-1}(w') = \text{span} \left\{ \begin{bmatrix} 0.1246 \\ 0.13101 \\ -0.0551 \end{bmatrix} \right\}$$

Inverse exists only if $\text{rank}(A) = 3$
$(\dim(\mathbb{R}^3))$

We know the $\text{colsp}(A)$ is given by the span of the column vectors, which is also the range of A, this is a subspace of the "target set" denoted by $\boxed{Im(A) = \text{colsp}(A)}$

A different subspace of the "domain" called the kernel of A is the set of all vectors $\vec{v} \in \mathbb{R}^n$ s.t. $A\vec{v} = 0$, in other words, the set of vectors in $\mathbb{R}^n$ mapped to zero vector in $\mathbb{R}^m$, These vectors are in the nullspace of A



$\boxed{\text{nullsp}(A) = \ker(A)}$

Rank and nullity of a linear mapping

$$\text{rank}(A) = \dim(\text{Im}\,A)$$
$$\text{nullity}(A) = \dim(\text{Ker}\,A)$$

Theorem: Let $\mathbb{R}^n$ be of finite dimension, and let $F_A : \mathbb{R}^n \to \mathbb{R}^m$, then

$$\dim(\mathbb{R}^n) = \dim(\ker A) + \dim(\text{Im}\,A)$$
$$= \text{nullity}(A) + \text{rank}(A)$$

Example

$$A = \begin{bmatrix} 5 & 1 & 0 \\ 4 & -1 & 1 \\ 3 & 0 & 8 \\ 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} \text{rank}(A) = 3 \\ \dim(\mathbb{R}^3) = 3 \end{array}$$

$$\dim(\ker A) = 0$$

only the zero vector maps to zero!!