Eigenvalues and Eigenvectors

Definition: Let A be any square matrix. A scalar λ is called an eigenvalue of A if there exists a non-zero (column) vector $\vec{v}$ s.t.

$$A\vec{v} = \lambda\vec{v}$$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \underbrace{\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\text{eigenvector}} \quad A\vec{v} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \underbrace{2}_{\text{eigenvalue}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} \stackrel{?}{=} \lambda\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

<u>not</u> an eigenvector

Eigenvectors give the direction of the charge under transformation and eigenvalues give the scalings

This transformation is applied to all vectors. Note, $\vec{v}$ is an invariant direction under the linear transformation

Eigenvectors are not unique

Consider $\vec{v} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = \underset{\text{eigenvalue}}{2} \underset{\text{eigenvector}}{\begin{bmatrix} 2 \\ 2 \end{bmatrix}}$$

Any scalar multiple of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is also an ~~eigenvalue~~ eigenvector with $\tau = 2$

$$A(k\vec{v}) = k(A\vec{v}) = k(\tau\vec{v}) = \tau(k\vec{v}) \therefore$$

$$A(\underbrace{k\vec{v}}_{\vec{u}}) = \tau(\underbrace{k\vec{v}}_{\vec{u}})$$

Calculating eigenvalues and eigenvectors

Example $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$

Step 1 Find the characteristic polynomial

$$p(s) = \det(sI - A)$$

$$sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} s-3 & -1 \\ 0 & s-3 \end{bmatrix}$$

$$p(s) = \begin{vmatrix} s-3 & -1 \\ 0 & s-3 \end{vmatrix} = \underline{\underline{(s-3)(s-3)}} - 0$$

Step 2 Calculate the roots to find the eigenvalues

$$p(\tau) = 0$$

$$(\tau_1 - 3)(\tau_2 - 3) = 0 \qquad (\tau - 3)(\tau - 2)$$

$$\Rightarrow \tau_1 = 3 \ , \ \tau_2 = 3$$

This eigenvalue has algebraic multiplicity of 2

Step 3 Find the eigenvectors

Find eigenvectors associated with $\tau = 3$

$$M = A - \tau I = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$(A - \tau I)\underset{\uparrow}{\vec{v}} = 0 \Rightarrow \boxed{M\vec{v} = 0}$$

eigenvector for $\tau = 3$

$$[M, \vec{b}] = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow$ $x_2 = 0$

x_1 Free variable

$$\vec{x} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$$ for any x_1 is an eigenvector

$E_{\tau=3} = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} : b = 0 \right\}$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is a basis for $E_{\tau=3}$

$\tau = 3$ has a geometric multiplicity of 1 because $\dim(E_{\tau=3}) = 1$

Example 2

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Compute eigenvalues by solving

$$\det(sI - A) = 0$$

$$sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} s-1 & -1 \\ -1 & s-1 \end{bmatrix}$$

$$\begin{aligned} \det(sI - A) &= (s-1)^2 - (-1)(-1) \\ &= (s-1)^2 - 1 \\ &= \tau^2 - 2\tau \cancel{+1} \cancel{-1} = 0 \\ &= \tau(\tau - 2) = 0 \end{aligned}$$

$$(s-1)^2 - 1 = 0$$

$$(s-1)^2 = 1 \Rightarrow s-1 = 1, \quad \Rightarrow s-1 = -1$$

$$\Rightarrow \tau = 0, \quad \tau = 2$$

each have algebraic multiplicity 1

Find eigenvector for $\tau = 0$

$$(A - \tau I)\vec{v} = 0$$

eigenvector

$\tau = 0$

$$(A - \tau I) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$[A, \vec{b}] = \left[\begin{array}{cc:c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cc:c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_1 + x_2 = 0$

x_2 : free variable

$x_1 = -x_2$

$$\vec{x} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad E_{\tau=0} = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} : a = -b \right\}$$

$$= \text{span}\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$$

geometric multiplicity
$= 1$

$\tau = 2$

$$\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{rref}([M, \vec{b}]) = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 - x_2 = 0$

$\Rightarrow x_1 = x_2$

$E_{\lambda=2} = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} : a = b \right\}$

geometric multiplicity 1

Example 3

$$A = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\det(sI - A) = \det\left(\begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 1 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)$$

$$\det\left(\begin{bmatrix} s-1 & 2 & 0 \\ 2 & s-1 & 0 \\ 0 & 0 & s-1 \end{bmatrix} \right)$$

$$\det(sI - A) = (s-1)(-1)^{(1+1)} \det\left(\begin{bmatrix} s-1 & 0 \\ 0 & s-1 \end{bmatrix} \right)$$

$$+ 2(-1)^{(1+2)} \det\left(\begin{bmatrix} 2 & 0 \\ 0 & s-1 \end{bmatrix} \right) + 0$$

$$= (s-1)^3 - 2(2)(s-1)) = 0$$

$$(s-1)\left(\underbrace{(s-1)^2 - 4}_{=0} \right) = 0$$

$$\boxed{T_1 = 1}$$

$$(s-1)^2 = 4$$

$$s-1 = 2 \Rightarrow \boxed{T_2 = 3}$$

$$s-1 = -2 \Rightarrow \boxed{T_3 = -1}$$

Eigenspace for $\tau_1 = 1$

$$A - \tau_1 I = \begin{bmatrix} 0 & -2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(A - \tau_1 I)\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{rref}([M, \vec{b}]) = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = 0$, $x_2 = 0$

x_3 : free variable

$$E_{\tau=1} = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a = b = 0 \right\}$$

<u>geometric multiplicity</u> of 1