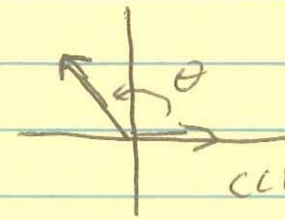
Lecture 12

Types of transformations $F_A : \mathbb{R}^n \to \mathbb{R}^n$

Reflection, rotation, dilation, translation

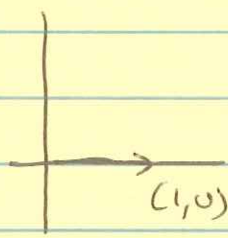
Rotation



CCW rotation

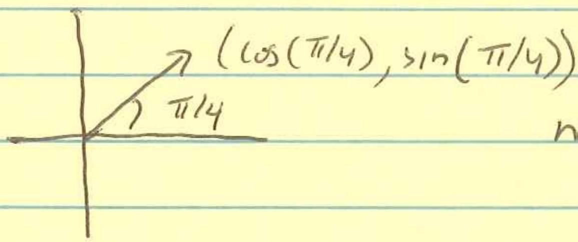
$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Let $\theta = 45°$ or $\pi/4$



$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} \cos(\pi/4) & -\sin(\pi/4) \\ \sin(\pi/4) & \cos(\pi/4) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos(\pi/4) \\ \sin(\pi/4) \end{bmatrix}$$



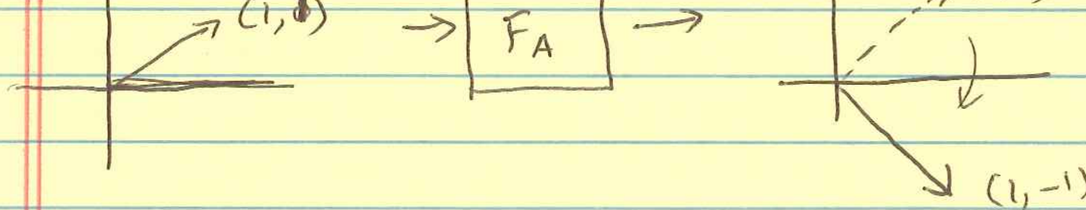
note that $|v| = 1$

Reflection

Example: reflect a vector across the x-axis

$$F_A \to \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Before



AR : rotation + reflection

$$AR = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos(\pi/4) & -\sin(\pi/4) \\ \sin(\pi/4) & \cos(\pi/4) \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & -\cos\theta \end{bmatrix}$$

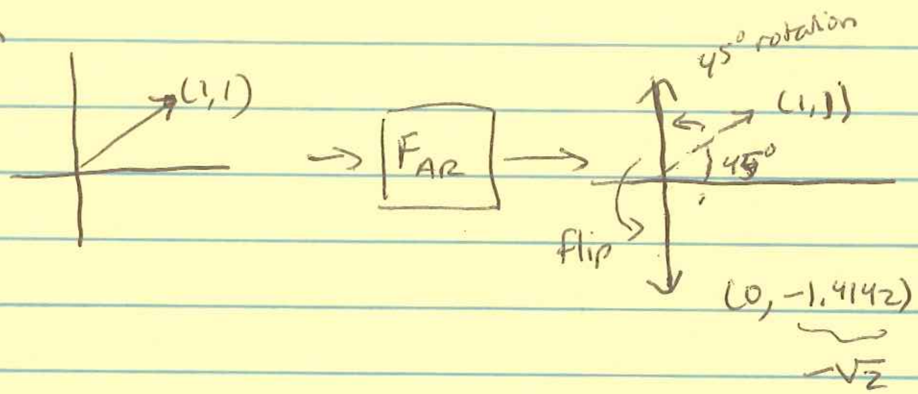
$$\vec{V} \to \boxed{F_{AR}} \to \vec{V} \text{ rotated + reflected}$$

$$AR \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta - \sin\theta \\ -\sin\theta - \cos\theta \end{bmatrix} \in \mathbb{R}^2$$

$\vec{V}_{in}$

$$\theta = \pi/4 \Rightarrow AR \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1.4142 \end{bmatrix} \leftarrow \vec{V}_{out}$$

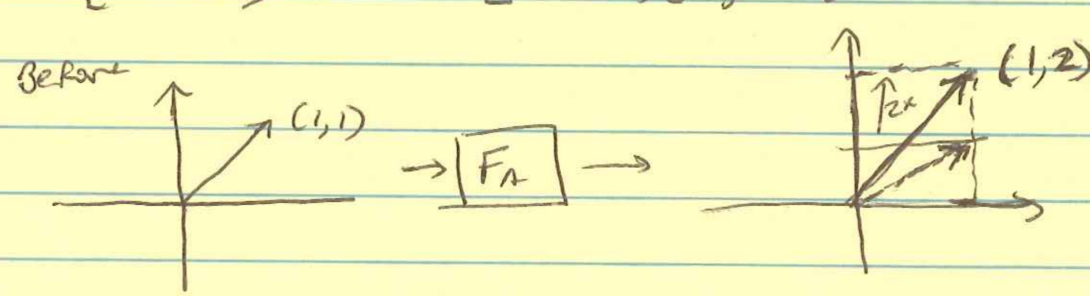
$\vec{V}_{in}$



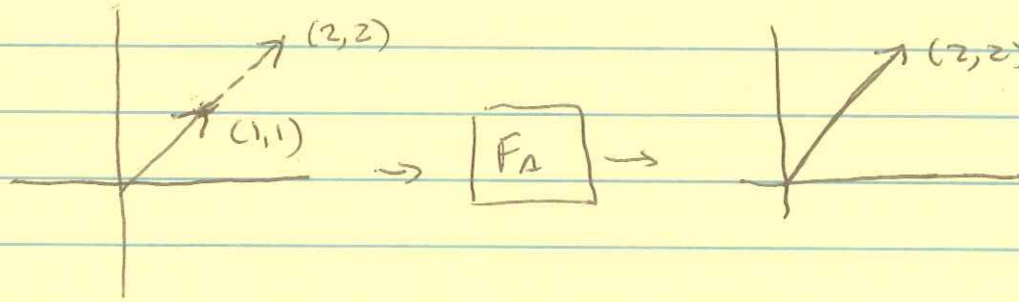
Dilation

Want to stretch my vector 2x in the y-direction

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$



What if I want to stretch the vector by 2x but along the direction of the vector



We require a matrix A s.t.

$$A\vec{v} = 2\vec{v}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \Rightarrow \begin{aligned} a\cdot 1 + b\cdot 1 &= 2 \\ c\cdot 1 + d\cdot 1 &= 2 \end{aligned}$$

one option is

$a = b = c = d = 1$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \checkmark$$

In this case we constructed a matrix A s.t. $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a <u>eigenvector</u> of matrix A with <u>eigenvalue</u> 2

Definition: Let A be any square matrix.
A scalar τ is called an eigenvalue of A if there exists a nonzero (column) vector $\vec{v}$ s.t.

$$Av = \tau v$$

Any vector satisfying this relation is called an eigenvector of A belonging to eigenvalue τ.

If we apply to difrerent vector

$$\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

Note that $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is not an eigenvector
"there does not exist" $\nexists$ scalar τ
s.t. $A\vec{v} = \tau \vec{v}$

Note that any scalar multiple of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is also an eigenvector with $\tau = 2$

$$A(k\vec{v}) = k(A\vec{v}) = k(\tau\vec{v}) = \tau(k\vec{v})$$

Ex: $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} \quad \tau = 2$$

$E_{\tau=2} = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ E_τ is the eigenspace of τ and is a subspace of the vector Field

Theorem A n-square matrix A is similar to a diagonal matrix D iff and only if A has n linearly independent eigenvectors. In this case, the diagonal elements of D are the corresponding eigenvalues and $D = P^{-1}AP$, where P is the matrix whose columns are the eigenvectors.