Lecture 12

Theorem A matrix $A \in \mathbb{R}^{n \times n}$ is similar to a diagonal matrix D iff (if and only if) A has n linearly independent eigenvectors.

$$D = \begin{bmatrix} \tau_1 & 0 & \cdots & 0 \\ 0 & \tau_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \tau_n \end{bmatrix}$$ Diagonal matrix

$D = P^{-1} A P$, where P is the matrix whose columns are the eigenvectors. P is a transition matrix ~~from stand~~ between two basis.

Coordinates and change of basis

$$\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\vec{v}$: coordinates relative to basis β

Basis $\beta = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

Consider the non-standard basis

$$\beta' \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right\}$$

$[\vec{v}]_{\beta'}$: coordinates relative to basis β'

$$[\vec{v}]_{\beta'} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

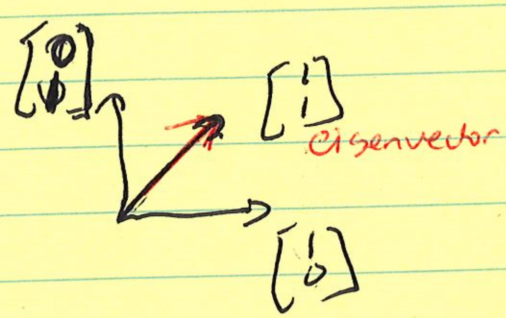
$$\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 1 \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

In $Ax=b$ format, we have again

$$\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Similarly, we can change basis in matrices

Diagonalization of matrix A is a change of coordinate basis that gives us a new perspective



$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

eigenvectors are $\underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\text{linearly independent}}$

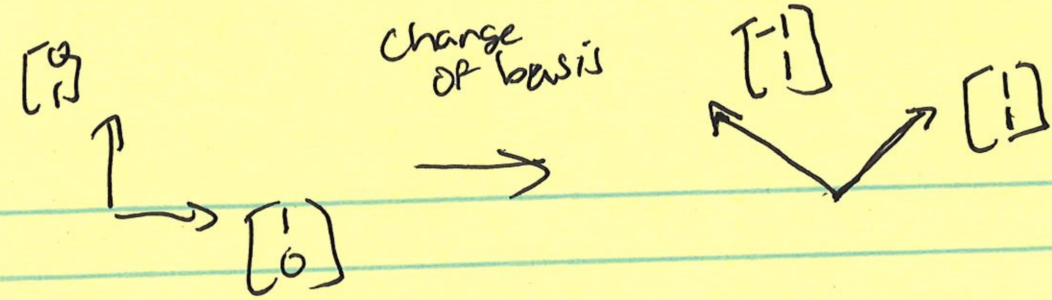
eigenvalues will be determined through diagonalization

$$P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}$$

$$D = P^{-1}AP = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{bmatrix}$$

$$\Rightarrow \tau_1 = 0, \quad \tau_2 = 2$$



A second derivation:

$$AP = A[\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n] = [A\vec{v}_1, A\vec{v}_2, \dots, A\vec{v}_n]$$

$$= [\tau_1 \vec{v}_1, \tau_2 \vec{v}_2, \dots, \tau_n \vec{v}_n]$$

$$= \underbrace{[\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n]}_{P} \begin{bmatrix} \tau_1 & & & \\ & \tau_2 & & \\ & & \ddots & \\ & & & \tau_n \end{bmatrix}$$

$$= PD$$

$$P^{-1}(AP = PD)$$

$AB \neq BA$

$$P^{-1}AP = \underbrace{P^{-1}P}_{I}D$$

$$\Rightarrow \boxed{D = P^{-1}AP}$$

$$\det(D) = \det(P^{-1}AP)$$

$$= \det(P^{-1})\det(A)\det(P)$$

$$= \det(P^{-1})\det(P)\det(A)$$

$$= \frac{1}{\cancel{\det(P)}} \cdot \cancel{\det(P)} \det(A)$$

$$= \det(A)$$

We have that $\boxed{\det(D) = \det(A)}$

$$\det\left(\begin{bmatrix} \tau_1 & 0 & 0 \\ 0 & \tau_2 & 0 \\ 0 & 0 & \tau_3 \end{bmatrix}\right) = \tau_1 \tau_2 \tau_3$$

$$\det(A) = \tau_1 \tau_2 \tau_3$$

$A \in \mathbb{R}^{2\times 2}$

$\text{rank}(A) = 1$

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

$$\det\left(\begin{bmatrix} s-1 & -1 \\ -2 & s-2 \end{bmatrix}\right)$$

$$= (s-1)(s-2) - 2$$

$$= s^2 - s - 2s + 2 - 2$$

$$= s^2 - 3s$$

$$= s(s-3) = 0$$

$$\Rightarrow \tau_1 = 0$$

$$\tau_2 = 3$$

$\det(A) = 1\cdot 2 - 1\cdot 2 = 0$ ✓

We must have a zero eigenvalue

To compute eigenvectors solve

$$(A - \tau I)\vec{v} = 0$$

$$\tau = 0$$

$$A\vec{v} = 0$$

eigenvectors are in the nullspace of kernel (A)

$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ $\quad$ $\text{rank}(A) = 1$

Revisit rank-nullity theorem

$A\vec{x}$

$\vec{x} \in \mathbb{R}^2$

$$\dim(\mathbb{R}^2) = \text{rank}(A) + \dim(\text{Ker}(A))$$
$$2 = 1 + 1$$

$$\text{rank}(A) = \dim(\text{colsp}(A))$$
$$= \dim(\text{Im}(A))$$
$$= \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$

Basis for $\text{Im}(A)$ is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Basis for nullspace

$\ast$ $A\vec{v} = \vec{0}$ $\quad$ solve for $\vec{v}$

Find eigenvectors associated with $\lambda = 0$

$$(A - \lambda I)\vec{v} = 0$$
$$\ast \quad A\vec{v} = 0$$

eigenvectors associated with $\lambda = 0$ is in the nullspace of A

$$\Rightarrow \dim(\text{Ker}(A)) \geq 1$$

Other benefits of diagonalization:

Computing powers of A

$$P(D = P^{-1}AP)P^{-1}$$

$$PDP^{-1} = \underbrace{PP^{-1}}_{I} A \underbrace{PP^{-1}}_{I}$$

$$\Rightarrow \quad A = PDP^{-1}$$

$$A^m = (PDP^{-1})^m$$

$$A^2 = (PDP^{-1})(PDP^{-1})$$
$$= PDDP^{-1} = PD^2P^{-1}$$

$$A^3 = A^2 A = (PD^2\underbrace{P^{-1})(P}_{I}DP^{-1})$$
$$= PD^3P^{-1}$$

$$A^m = PD^mP^{-1}$$

$$D = \begin{bmatrix} d_1 & & \\ & d_2 & \\ & & d_3 \end{bmatrix} \Rightarrow D^m = \begin{bmatrix} d_1^m & & \\ & d_2^m & \\ & & d_3^m \end{bmatrix}$$

Computing eigenvalues for large upper / lower triangular matrices

upper triangular matrix

$$A = \begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 7 & -1 & 2 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 7 & 0 & 0 \\ 4 & -1 & 4 & 0 \\ 5 & 2 & 0 & -1 \end{bmatrix}$$

lower triangular matrix

$$\det(sI - A) = \det(sI - A^T)$$

eigenvalues of A = eigenvalues of A^T

$$\det\left(\begin{bmatrix} s-1 & 0 & 0 & 0 \\ -2 & s-7 & 0 & 0 \\ -4 & 1 & s-4 & 0 \\ -5 & -2 & 0 & s+1 \end{bmatrix} \right)$$

$$= (s-1)(s-7)(s-4)(s+1)$$

eigenvalues ~~the dia~~ are the diagonal entries

$$\tau_1 = 1, \quad \tau_2 = 7, \quad \tau_3 = 4, \quad \tau_4 = -1$$

For each τ_i

$$(A - \tau_i I)\vec{v} = 0$$

Find augmented matrix $[A - \tau_i I, \vec{0}]$

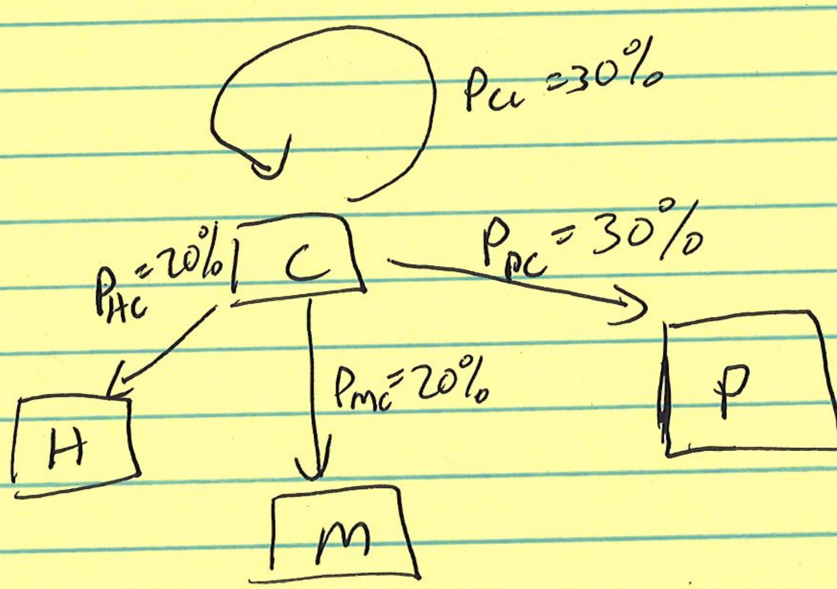
$\text{rref}[A - \tau_i I]$

Applications of Eigenvalues

Markovian Process

Consider 4 dinner options

C Chinese
M Mexican
P Pizza
H Home



We construct a transition matrix

$$
A = \begin{array}{c} \\ \\ \\ \\ \\ \end{array}
\begin{array}{cccc|c}
H & C & M & P & \\
\end{array}
$$

$$
\begin{array}{c} \\ A = \end{array}
\begin{array}{cccc}
H & C & M & P \\
\end{array}
$$

Probability of eating at home after eating at home

Probability of eating Mexican after eating Chinese

$[A]_{ij} = a_{ij}$ the probability of state i after I was in state j

Day 1 Suppose IC 100% probability H

$$X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$ probability distribution

Day 2

$$x_2 = A x_1$$

Day 3 $x_3 = A x_2 = A^2 x_1$

Day m $x_m = A^m x_1$