12 Week 5 Wednesday

We have explored different methods finding the Green’s function including applying transforms and the divergence theorem (for the infinite space Laplacian) and in your most recent HW, series representation using the eigenfunction.

We have learned about Fourier transforms, Fourier sine and cosine transforms, Laplace transforms and there exists many more. Recall, these are all methods used to make our lives easier especially when tasked with finding a Green’s function.

We learned that the Green’s function is defined by the response of a system to an impulse forcing function. Due to the linear nature of the PDEs we have encountered, we can use the Green’s function to find the response to any forcing function through convolution. For the transforms we have looked at, note that convolution in the time and space domain was converted to product in the frequency domain. We refer to the transformed domain as the frequency domain since the "basis" is composed of signals of all frequencies.

We first looked at 1D equations and showed that the Green’s function for the heat equation

\[ g_t - kg_{xx} = \delta(x - x_0)\delta(t - t_0) \]

is given by

\[ g(x, t; x_0, t_0) = \frac{1}{\sqrt{4\pi k(t - t_0)}} e^{-(x-x_0)^2/(4\kappa(t-t_0))} \]

which we were then able to use this to apply to the general problem

\[ u_t - ku_{xx} = Q(x, t) \]  \hspace{1cm} (398)
\[ u(0, t) = A(t) \]  \hspace{1cm} (399)
\[ u(x, 0) = f(x) \]  \hspace{1cm} (400)

We first found an equivalent system with homogeneous BCs through the transformation \( u = w(x, t) - A(t) \) and solved the equivalent problem

\[ u_t - ku_{xx} = \tilde{Q}(x, t) \]  \hspace{1cm} (401)
\[ u(0, t) = 0 \]  \hspace{1cm} (402)
\[ u(x, 0) = \tilde{f}(x) \]  \hspace{1cm} (403)

for which the solution is

\[ u(x, t) = \int_{-\infty}^{\infty} \tilde{f}(\xi)g(x - \xi, t)d\xi + \int_{0}^{t} \int_{-\infty}^{\infty} p(\xi, \tau)g(x - \xi, t - \tau)d\xi d\tau. \]

We also demonstrated the utility of transform on their own to find solutions without Green’s function and looked at the wave equation in 1D

\[ u_{tt} = c^2 u_{xx} \]

where we derived the d’Alambert’s solution using the Fourier transform.

We then looked at how we could apply the method of images in order to to look at problems with semi-infinite domain and even finite domains. In this case we extended the system to infinite space and added solutions to opposing impulses in order to satisfy boundary conditions.
Combining Fourier transform in space and Laplace transform in time allowed us to solve some more complicated equations such as the 1D heat transfer with convection
\[ u_t - \kappa u_{xx} - cu_x = \delta(x - \xi)\delta(t - \tau) \]
and the wave equation with sinusoidal pumping
\[ u_{tt} - c^2 u_{xx} = \delta(x) \sin(\omega_0 t). \]

Finally, we moved on to 2D problems for the first time in the class. We solved the Laplacian \((\nabla^2 = 0)\) in the rectangle (separation of variables) and the semi-infinite strip (Fourier sine transform in \(x\)) with arbitrary boundary conditions. Note that these are time-invariant problems. We were also able to solve them without the Green’s function method. Up until now, we hadn’t used Green’s function in 2D for finite domains.

The natural next question was how to find Green’s function for the Laplacian, which would then allow us to solve problems with forcing functions and BCs on an arbitrary domain.

We first solved the \(\mathbb{R}^2\) free space Green’s function
\[ g(x, \xi) = \frac{1}{2\pi} \ln(|x - \xi|) \]
using the divergence theorem for the Laplacian
\[ \nabla^2 g(x, \xi) = \delta(x - \xi) \]
which we can use to find a solution to the inhomogeneous problem
\[ \nabla^2 u(x) = q(x) \]
which is the "sum" over the impulse response at every point in space
\[ u(x, y) = \frac{1}{2\pi} \int_\infty^-\infty \int_\infty^-\infty \ln(|\tilde{x} - \tilde{\xi}|)q(\tilde{\xi})d\xi_1d\xi_2. \]
This solution is for the infinite domain.

We then introduced the Helmholtz equation in 2D
\[ \nabla^2 u - \lambda u = q, q = q(\tilde{x}), \]
\[ u|_{\partial \Omega} = f(x), \tilde{x} \in \mathbb{R}^2 \text{ or } \mathbb{R}^3 \]
which is a bit of a different beast because we have a boundary condition on a two-dimensional space. So we can find the Green’s function but it is unclear how to relate this to an inhomogeneous problem. Using Green’s theorem we derived this exact expression
\[ u(x) = \int_R G(x, a)q(a)da + \int_{\partial R} f(a) \frac{\partial G}{\partial n_x}(x, a)dS \text{ for } x \in \mathbb{R}^2 \text{ or } \mathbb{R}^3 \]

Last lecture we ended with the solution to the Green’s function
\[ G(x, y; 0, 0) = -\frac{1}{4\pi^2} \int_0^{2\pi} \int_0^\infty \frac{e^{i\rho \cos(\theta)} d\rho d\theta}{\rho^2 + \lambda}. \]

Using some known integrals
\[ \int_0^\pi e^{i\beta \cos(x)} dx = \pi J_0(\beta) \]
and
\[
\int_0^{\infty} \frac{x J_0(ax)}{x^2 + k^2} dx = K_0(ak)
\]
we can simplify the solution to
\[
G(x, y; 0, 0) = -\frac{1}{2\pi} K_0(\sqrt{x^2 + y^2})
\]

### 12.1 Example

We now examine a problem where this solution comes in handy. Consider the 2D wave equation
\[
w_{tt} - c^2 \nabla^2 w = \delta(x - \xi) e^{-i\omega t}.
\]
We propose a solution of the form
\[
w(x, t) = e^{-i\omega t} \phi(x)
\]
and plugging into the ODE we get
\[
e^{-i\omega t} (-\omega^2 \phi - c^2 \nabla^2 \phi) = \delta(x - \xi) e^{-i\omega t}
\]
\[
\Rightarrow \nabla^2 \phi + \frac{\omega^2}{c^2} \phi = -\frac{\delta(x - \xi)}{c^2}
\]
and this looks just like the setup for the Green’s function we just solved with a gain of $1/c^2$ so the solution is
\[
\phi(x) = \frac{1}{2\pi c^2} K_0(\pm \frac{\omega}{c} |x|)
\]
\[
K_0(z) \approx \left\{ \begin{array}{ll}
\sqrt{\frac{\pi}{2z}} e^{-z} & z \to \infty \\
-\ln \left( \frac{\sqrt{z}}{2} \right) & z \to 0
\end{array} \right.
\]
Note that the solution resembles the free space solution to the Laplacian as $x \to 0$, which is the same as taking $\lambda \to 0$, in which case we get back the Laplacian. As $x \to \infty$, our solution is
\[
\phi(x) = \frac{1}{2\pi c^2} \sqrt{\frac{\pi}{2(\pm i\omega |x|/c)}} e^{\mp i\omega |x|/c}
\]
and the complete solution including the time-dependent portion is
\[
w(x, t) = \frac{1}{2\pi c^2} \sqrt{\frac{\pi}{2(\pm i\omega |x|/c)}} e^{-i\omega(t \mp |x|/c)}.
\]
The physical solution pertains to the real part
\[
\phi(x, \xi) = \frac{1}{2\pi c^2} \sqrt{\frac{\pi c}{2(\pm \omega |x|)}} \cos(\omega(t \mp |x|/c) - \pi/4)
\]
but it doesn’t make sense for waves to move in from infinity so only the minus sign survives
\[
\phi(x, \xi) = \frac{1}{2\pi c^2} \sqrt{\frac{\pi c}{2|\omega| |x|}} \cos(\omega(t - |x|/c) - \pi/4).
\]
12.2 Multidimensional Eigenfunction expansion

In your last HW, you were asked to express Green’s function as a sum of a eigenfunctions. For S-L eigenvalue problems in finite domain, one can express the Green’s function as a series summation of eigenfunctions. We construct the Green’s function for Laplace’s equation. Write

\[ G(x, x_0) = \sum_n a_n \phi_n(x) \]

where \( \phi_n \) is the eigenfunction corresponding to the eigenvalue \( \lambda_n \). Then we apply the Laplacian operator so that

\[ \nabla^2 G(x, x_0) = \sum_n a_n \nabla^2 \phi_n(x) = -\sum_n a_n \lambda_n \phi(x) \]

Since the left hand side is \( \delta(x - x_0) \), by orthogonality and properties of the delta function, we get

\[ -a_n \lambda_n = \frac{\phi_n(x_0)}{\int_D \phi^2_n(x) dx} \]

12.3 Method of eigenfunction expansion on a rectangular domain

Consider the two dimensional problem

\[ \phi_{xx} + \phi_{yy} + \lambda \phi = 0 \]
\[ \phi(x, 0) = 0 \]
\[ \phi(x, H) = 0 \]
\[ \phi(L, y) = 0 \]
\[ \phi(0, y) = 0 \]

with homogeneous boundary conditions. The two dimensional eigenvalues are

\[ \phi_{mn}(x, y) = \sin \left( \frac{n\pi x}{L} \right) \sin \left( \frac{m\pi y}{H} \right) \]

for \( m, n = 1, 2, 3 \ldots \) with associated eigenvalues

\[ \lambda_{mn} = \left( \frac{n\pi}{L} \right)^2 + \left( \frac{m\pi}{H} \right)^2. \]

In this case the eigenfunction expansion of the Green’s function is

\[ G(x, x_0) = -\frac{4}{ LH} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\sin(n\pi x/L) \sin(m\pi x/H) \sin(n\pi x_0/L) \sin(m\pi y_0/H)}{(n\pi/L)^2 + (m\pi/H)^2} \]