Lecture 13

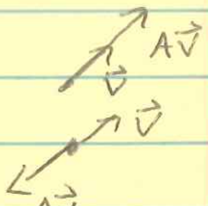
Last week we introduced the concept of eigenvectors and eigenvalues. That is if there exists a nonzero vector $\vec{v}$ and scalar τ s.t.

$$A\vec{v} = \tau\vec{v}$$

then τ is an eigenvalue with associated eigenvector $\vec{v}$.

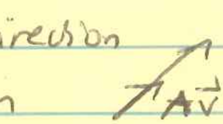
$\vec{v}$ is an invariant direction under the linear transformation

If ..



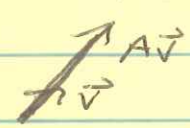
$\tau > 0$ $\vec{v}$ and $A\vec{v}$ have the same direction

$\tau < 0$ " " " " " opposite direction



$|\tau| < 1$ eigenvalue is a factor of contraction

$|\tau| > 1$ " " extension



If an $n \times n$ square matrix A has n linearly independent eigenvectors then there exists a transition matrix P (lecture 11) s.t.

$$P^{-1}AP = \begin{bmatrix} \tau_1 & 0 & \cdots & 0 \\ 0 & \tau_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \tau_n \end{bmatrix}$$

that is there exists a basis representation for which the transformation matrix A is a diagonal matrix.

P is the transition matrix from a basis composed of eigenvectors to the standard basis

Let
$$B = \{e_1, e_2, \ldots, e_n\} \text{ and } B' = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$$

standard basis — n linearly independent eigenvectors

From Lecture 11
$$[\vec{v}]_B = P\,[\vec{v}]_{B'}$$

Then P is composed of the eigenvectors
$$P = [\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n]$$

$$AP = A[\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n] = [A\vec{v}_1, A\vec{v}_2, \ldots, A\vec{v}_n]$$
$$= [\tau_1\vec{v}_1, \tau_2\vec{v}_2, \ldots, \tau_n\vec{v}_n]$$
$$= \underbrace{[\vec{v}_1, \vec{v}_2, \ldots, \vec{v}]}_{P}\begin{bmatrix}\tau_1 & & & \\ & \tau_2 & & \\ & & \ddots & \\ & & & \tau_n\end{bmatrix}$$
$$= PD$$

$$\boxed{D = P^{-1}AP}$$

In the diagonalized matrix the coordinate axis' line up with the eigenvectors. A ~~trans~~ transformation along one of the axis results in a vector along the same axis.

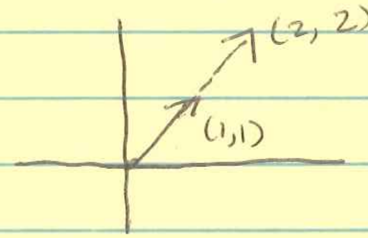
$$\det(D) = \det(P^{-1}AP)$$
$$= \det(P^{-1})\det(A)\det(P)$$
$$= \det(P^{-1}) \cdot \det(P)\det(A)$$
$$= \frac{1}{\det(P)} \cdot \det(P)\det(A)$$
$$= \det(A)$$

$$\det(D) = \lambda_1 \lambda_2 \lambda_3 \ldots \lambda_n$$

$\det(A)$ is equal to the product of the eigenvalues

In Thursday's Lecture we explored the example where we had a matrix with eigenvalue 2 and associated eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$



$$A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

Propose a second eigenvector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$$

$A\vec{v} = \lambda\vec{v}$

where $\lambda = 0$

So A has two linearly independent eigenvectors

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Note that A does <u>not</u> necessarily have linearly indep. column/row vectors!

Our theorem says we can construct a matrix P from the eigenvectors

$$P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

and that $P^{-1}AP$ will give us a matrix with the eigenvalues along the diagonal.

$$D = P^{-1}AP = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \qquad \lambda_1 = 0 \ \checkmark \qquad \lambda_2 = 2 \ \checkmark$$

A is called <u>diagonalizable</u>.

$$P\left(D = P^{-1}AP\right)P^{-1}$$

$$PDP^{-1} = \underbrace{PP^{-1}}_{I} A \underbrace{PP^{-1}}_{I}$$

$$\Rightarrow \boxed{A = PDP^{-1}}$$

$$A^m = (PDP^{-1})^m = P \begin{bmatrix} T_1^m & & & \\ & T_2^m & & \\ & & \ddots & \\ & & & T_n^m \end{bmatrix} P^{-1}$$

$$A^2 = (PDP^{-1})^2$$
$$= (PD\underbrace{P^{-1})(P}_{I}DP^{-1})$$
$$= PD^2P^{-1}$$
$$A^3 = A^2A = (PD^2\underbrace{P^{-1})(P}_{I}DP^{-1})$$
$$= PD^3P^{-1}$$
$$\vdots$$
$$A^m = PD^mP^{-1}$$

For a diagonal matrix, we have

$$D = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix} \Rightarrow D^m = \begin{bmatrix} d_1^m & & & \\ & d_2^m & & \\ & & \ddots & \\ & & & d_n^m \end{bmatrix}$$

which one can verify through induction as well.

Properties of eigenvalues and eigenvectors

Theorem: Let A be a square matrix. Then the following are equivalent

(i) A scalar λ is an eigenvalue of A

(ii) The matrix $M = A - \lambda I$ is singular
(singular meaning $\det(M) = 0$ or equiv. column vectors are linearly dependent)

(iii) The scalar λ is a root of the characteristic polynomial $p(s)$ of A
$p(s) = \det(sI - A)$

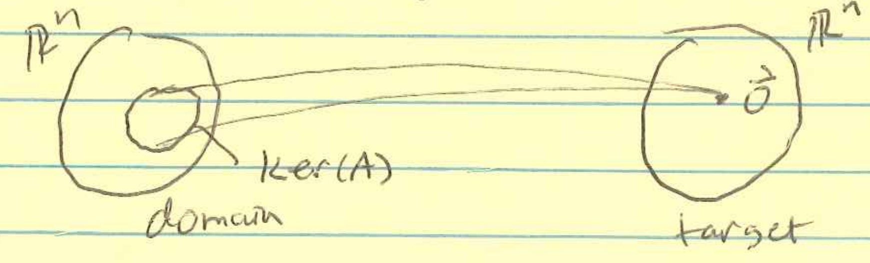
A closer look at (ii)

Singular and nonsingular linear mappings

Linear matrix mapping
$F_A : \mathbb{R}^n \to \mathbb{R}^n$, where $F_A(\vec{v}) = A\vec{v}$

The mapping F_A is said to be singular if the image of some nonzero vector $\vec{v}$ is $\vec{0}$, that is, if $\exists$ ("there exists") $v \neq 0$ s.t. $F_A(\vec{v}) = \vec{0}$ $(A\vec{v} = \vec{0})$.

Thus $F_A : \mathbb{R}^n \to \mathbb{R}^n$ is nonsingular if the zero vector $\vec{0}$ is the only vector whose image under F_A is $\vec{0}$
$(A\vec{v} = 0$ iff $\vec{v} = 0)$, in other words, if $\ker A = \{\vec{0}\}$



Note that if $\ker A = \{0\}$ then
$\text{rank}(A) = n$ the matrix is <u>full rank</u>

$$\dim(\mathbb{R}^n) = \text{rank}(A)$$

$$\left(\dim(\ker(A))\right) = 0$$

(ii) "is saying" that the matrix
$M = A - \tau I$ is guaranteed to have
a set of vectors (nonzero) that
map to the zero vector.

The set of vectors include the
eigenvectors associated with eigenvalue τ

$$A\vec{v} = \tau\vec{v}$$

$$A\vec{v} - \tau\vec{v} = 0$$

$$\underbrace{(A - \tau I)}_{M}\vec{v} = 0$$

$$M\vec{v}$$

$$\therefore \quad \vec{v} \in \ker(A - \tau I) \qquad I \in \mathbb{R}^{n \times n}$$

$n \times n$ identity matrix

(iii) Characteristic polynomial $p(s)$ is as defined
$p(s) = \det(sI - A)$

Consider 2×2 matrix $A = \begin{bmatrix} 1 & -2 \\ 0 & 4 \end{bmatrix}$

then $sI - A = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -2 \\ 0 & 4 \end{bmatrix}$

$$= \begin{bmatrix} s-1 & 2 \\ 0 & s-4 \end{bmatrix}$$

$$\det(sI-A) = (s-1)(s-4) - 0\cdot 2$$
$$= s^2 - 5s + 4$$

$$\Rightarrow p(s) = s^2 - 5s + 4$$

(iii) "is saying"

$p(\tau) = 0$ τ is a root of $p(s)$

$$\Rightarrow \tau^2 - 5\tau + 4 = 0$$

(side note: It is also true that $p(A) = 0$)

$$\begin{bmatrix} 1 & -2 \\ 0 & 4 \end{bmatrix}^2 - 5\begin{bmatrix} 1 & -2 \\ 0 & 4 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Cayley-Hamilton theorem: Every matrix A is a root of its characteristic polynomial

(i), (ii), (iii) are properties that help us to find eigenvalues and eigenvectors and, hence, factorized representation of $A \in \mathbb{R}^{n\times n}$