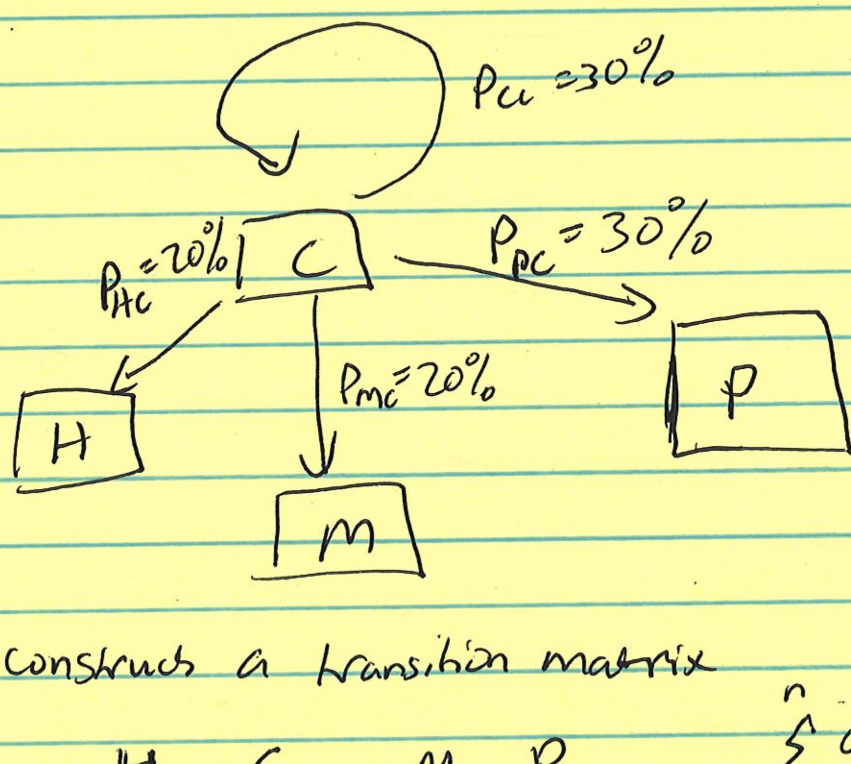
Applications of Eigenvalues

Markovian Process

Consider 4 dinner options

C Chinese
M Mexican
P Pizza
H Home



We construct a transition matrix

$$A = \begin{array}{c} \\ \\ \\ \\ \end{array}\begin{array}{cccc} H & C & M & P \\ \end{array}$$

$$A = \begin{bmatrix} .25 & .2 & .25 & .3 \\ .20 & .3 & .25 & .3 \\ .25 & .2 & .4 & .1 \\ .30 & .3 & .1 & .3 \end{bmatrix} \begin{array}{c} H \\ C \\ M \\ P \end{array}$$

$$\sum_{i=1}^{n} a_{ij} = 1 \quad j \text{ fixed}$$

Probability of eating at home after eating at home

Probability of eating Mexican after eating Chinese

$[A]_{ij} = a_{ij}$ the probability of state i after I was in state j

Day 1 Suppose IC 100% probability H

$$X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$ probability distribution

Day 2

$$X_2 = AX_1 = \begin{bmatrix} .25 \\ .20 \\ .25 \\ .30 \end{bmatrix}$$

Day 3 $$X_3 = AX_2 = A^2 X_1 = \begin{bmatrix} .2550 \\ .2625 \\ .2325 \\ .25 \end{bmatrix}$$

Day m $$X_m = A^m X_1$$

Day 30

$$X_{30} = A^{30} X_1 = \begin{bmatrix} .2495 \\ .2634 \\ .2339 \\ .2532 \end{bmatrix}$$

Day 50

$$X_{50} = A^{50} X_1 = \begin{bmatrix} .2495 \\ .2634 \\ .2339 \\ .2532 \end{bmatrix}$$ Same!

$$\lim_{m \to \infty} X_m = X^*$$ where * denotes our equilibrium state

X^* captures the probability of eating at any restaurant over a long period of time

Why can we guarantee convergence?

Perron - Frobenius theorem

Let $A \in \mathbb{R}^{n \times n}$ be a positive square matrix. All entries $a_{ij} > 0$. Then the largest eigenvalue has algebraic multiplicity 1 and is a positive real number.

For a transition matrix (stochastic matrix) that eigenvalue is 1.

$$1 = \tau_1 > |\tau_2| > |\tau_3| > \ldots > |\tau_n|$$

Assume A is diagonalizable (not always true)

$$A^m = (PDP^{-1})^m = PD^mP^{-1}$$

$$= P \begin{bmatrix} \tau_1 = 1^m & & & \\ & \tau_2^m & & \\ & & \ddots & \\ & & & \tau_n^m \end{bmatrix} P^{-1}$$

if $|\tau_i| < 1$

$$\lim_{m \to \infty} \tau_i^m = 0$$

Recall, that A is diagonalizable if and only if its eigenvectors are linearly indep.

$A \in \mathbb{R}^{n \times n} \Rightarrow$ n linearly independent eigenvectors in $\mathbb{R}^n$

$$\operatorname{span}\{\underbrace{\vec{v}_1}_{\text{eigenvectors}}, \dots, \vec{v}_n\} = \mathbb{R}^n$$

Then

$$\lim_{m \to \infty} A^m x = c_1 \overset{1}{\lambda_1} v_1 + c_2 \overset{0}{\lambda_2} v_2 + \dots + c_n \overset{0}{\lambda_n} v_n$$

$$= c_1 1 v_1$$

X^*: equilibrium state is ~~the~~ an eigenvector associated with $\lambda = 1$

Such guarantees cannot be made if there are zero entries $a_{ij} \geq 0$

Example
$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Lets compute eigenvalues!

$$sI - A = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ -1 & 0 & s \end{bmatrix}$$

$$\det(sI-A) = s \begin{vmatrix} s & -1 \\ 0 & s \end{vmatrix} + 1 \begin{vmatrix} 0 & -1 \\ -1 & s \end{vmatrix}$$

$$= s^3 - 1 = 0$$

$$s^3 = 1 = e^{i2\pi k}$$

$$= \cos(2\pi k) + i\sin(2\pi k)$$

$\Rightarrow$ 3 eigenvalues

$$\left(e^{i2\pi k}\right)^{1/3}$$

$$s = e^{i\frac{2}{3}\pi k}$$

$k=1$ $\quad \tau_1 = e^{i2\pi/3} = \cos(2\pi/3) + i\sin(2\pi/3)$

$k=2$ $\quad \tau_2 = e^{i4/3\pi} = \cos(4/3\pi) + i\sin(\frac{4}{3}\pi)$

$k=3$ $\quad \tau_3 = e^{i2\pi} = 1$

$$|\tau_1| = |\tau_2| = |\tau_3| = 1$$

periodic Markov Chain

$$A^{100} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$A^{101} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$A^{102} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Complex eigenvalues / eigenvectors

$$A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

$$\det(sI - A) = \det\left(\begin{bmatrix} s & -2 \\ 2 & s \end{bmatrix}\right) = s^2 + 4 = 0$$

$$s^2 = -4$$

$$s = \pm\sqrt{-4}$$

$$= \pm i2$$

$$\boxed{T_1 = 2i} \quad \boxed{T_2 = -2i}$$

$T_1 = 2i$

$$(A - T_1 I)x = 0$$

$$\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} - \begin{bmatrix} 2i & 0 \\ 0 & 2i \end{bmatrix} = \begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix}$$

$$L_1(i) + L_2 \to L_2 \qquad \begin{bmatrix} -2i & 2 \\ 0 & 0 \end{bmatrix}$$

$$-2i\,x_1 + 2x_2 = 0 \qquad 2x_2 = 2ix_1$$

$$\Rightarrow x_1 = -ix_2$$

$$x = x_2 \begin{bmatrix} -i \\ 1 \end{bmatrix} \qquad E_{T=2i} = \operatorname{span}\left\{ \begin{bmatrix} -i \\ 1 \end{bmatrix} \right\}$$

$\lambda_2 = -2i$

$$A - \lambda_2 I = \begin{bmatrix} 2i & 2 \\ -2 & 2i \end{bmatrix}$$

$L_1(-i) + L_2 \to L_2$

$$\begin{bmatrix} 2i & 2 \\ 0 & 0 \end{bmatrix}$$

$$\cancel{2}i\, x_1 + \cancel{2} x_2 = 0$$

$$(ix = -x_2)\, i$$

$$+x = +ix_2$$

$$E_{\lambda = -2i} = \text{span}\left\{ \begin{bmatrix} i \\ 1 \end{bmatrix} \right\}$$

Examples of diagonalizable / non-diagonalizable matrix

Case 1

$$A = \begin{bmatrix} 1 & 5 & 3 & 1 \\ 0 & 2 & 4 & 5 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 7 \end{bmatrix}$$

eigenvalues $\lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 3,\ \lambda_4 = 7$

4 distinct eigenvalues $\Rightarrow$ A is diagonalizable

(diagonalizable A does not imply distinct eigenvalues)

Case 2

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 4 \\ 0 & 0 & 2 \end{bmatrix} \qquad \begin{bmatrix} s-2 & 0 & 0 \\ -1 & s-3 & -4 \\ 0 & 0 & s-2 \end{bmatrix}$$

$$\det(sI - A) = (s-2)^2 (s-3) = (s-2)(s-2)(s-3)$$

$\lambda_1 = 2$ algebraic multiplicity

$\lambda_2 = 2$

$\lambda_3 = 3$

For diagonalizability check geometric multiplicity for $\lambda = 2$

$(A - \lambda I)\vec{x} = 0$

$$A - 2 \cdot I = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = -x_2 - 4x_3$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 - 4x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$$

basis for $E_{\lambda=2}$ $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\lambda = 2$ has geometric multiplicity of 2
geometric multiplicity equals my
algebraic multiplicity
$\Rightarrow$ A <u>is diagonalizable</u>

Case 3

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 4 \\ 1 & 0 & 2 \end{bmatrix} \Rightarrow \begin{matrix} \lambda_1 = 2 \\ \lambda_2 = 2 \\ \lambda_3 = 3 \end{matrix}$$

$\lambda = 2$

$$A - 2I = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 4 \\ 1 & 0 & 0 \end{bmatrix} \xrightarrow[\text{rref}]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 0 \qquad x_2 + 4x_3 = 0$ ← 1 free variable

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -4x_3 \\ x_3 \end{bmatrix} = x_3 \underbrace{\begin{bmatrix} 0 \\ -4 \\ 1 \end{bmatrix}}_{\text{basis for } E_{T=2}}$$

geometric multiplicity of $\underline{1}$

$\Rightarrow$ A is <u>not</u> diagonalizable

Singular Value Decomposition (SVD)

Factorization for non-diagonalizable matrices (square, non-square)

Factorize matrix as follows

$$A = U \Sigma V^T \qquad \left(\overset{\text{diagonalizable}}{A = PDP^{-1}}\right)$$

U composed of the left singular vectors of A

$\Rightarrow$ orthonormal eigenvectors of AA^T (square matrix)

V compose of the right singular vectors of A

$\Rightarrow$ orthonormal eigenvectors of A^TA (square matrix)

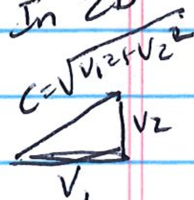
Orthonormal : ~~normal~~ normalized (unit) and orthogonal vectors

norm~~al~~ equal to 1 $\boxed{\|\vec{v}\| = 1}$

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2} = 1$$

In 2D



$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$, $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

$$a \cdot b = a_1 b_1 + a_2 b_2 + \dots + b_n a_n$$
$$= a^T b$$
$$= b^T a$$

$$a^T = [a_1 \ a_2 \ \dots \ a_n]$$
$$b^T = [b_1 \ b_2 \ \dots \ b_n]$$

Orthogonal

$$\vec{a} \perp \vec{b} \iff a \cdot b = 0$$

Example: