Lecture 14 Calculating eigenvalues and eigenvectors

Example $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Step 1 Find the characteristic polynomial

$$p(s) = \det(sI - A)$$

$$sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} s-1 & -1 \\ -1 & s-1 \end{bmatrix}$$

$$p(s) = \det(sI - A) = (s-1)^2 - (-1)(-1)$$
$$= (s-1)^2 - 1$$

Step 2 Calculate the roots to find the eigenvalues

$$p(\tau) = 0 \qquad \tau^2 - 2\tau + \cancel{1} - \cancel{1} = 0$$
$$\tau(\tau - 2) = 0$$
$$\Rightarrow \tau = 0, \quad \tau = 2 \ ✓$$

Step 3 Find the eigenvectors

$$M = A - \tau I$$

$$(A - \tau I)\vec{v} = 0$$

eigenvector associated with eigenvalue τ

$$\vec{v} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\tau = 0$

$$\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\vec{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $[A, \vec{b}]$ $[M, \vec{b}]$ $= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

$$\text{rref}([A, \vec{b}]) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad x_1 + x_2 = 0$$
$$x_1 = -x_2 \qquad \vec{v} = \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix}$$

eigenvector space for $\tau = 0$

$$E_{\tau=0} = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} : a = -b \right\}$$

$\tau = 2 \qquad (\underbrace{\cancel{A} - 2I}_{M})\vec{v} = 0$

$$\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{rref}([M, b]) = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad x_1 - x_2 = 0 \qquad \vec{v} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix}$$
$$x_1 = x_2$$

$$E_{\tau=2} = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} : a = b \right\}$$

Step 4 Consider the collection

$S = \{ v_1, v_2, \dots, v_m \}$ of

all eigenvalues obtained in step 3

(a) If $m \neq n$ (where $A \in \mathbb{R}^{n \times n}$) then A is not diagonalizable

(b) If $m = n$, the A is diagonalizable

Example 2 $\quad A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$

$$sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} s-3 & -1 \\ 0 & s-3 \end{bmatrix}$$

$$\det(sI - A) = (s-3)^2$$

$$p(\tau) = 0 \Rightarrow (\tau - 3)^2 = 0 \quad [(\tau-3)(\tau-3) = 0]$$
$$\Rightarrow \tau = 3$$

Eigenvalue of algebraic multiplicity 2

$$(\underbrace{A - (3)I}_{M})\vec{x} = 0$$

$$\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$[M, b] = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow$

$x_2 = 0$

x_1 free variable

$$\vec{x} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$$

$$E_{T=3} = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} : b = 0 \right\}$$

Basis has one element, geometric multiplicity of 1

Example 3

$$A = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\det(sI - A) = \det\left(\begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 1 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)$$

$$\equiv \det(A - sI)$$

$$\det(A - \tau I)$$

$$= \det\left(\begin{bmatrix} s-1 & 2 & 0 \\ 2 & s-1 & 0 \\ 0 & 0 & s-1 \end{bmatrix} \right)$$

$$\det(sI-A) = (s-1)(-1)^{(1+1)} \det \begin{bmatrix} s-1 & 0 \\ 0 & s-1 \end{bmatrix} + 2(-1)^{(1+2)} \det \begin{bmatrix} 2 & 0 \\ 0 & s-1 \end{bmatrix} + 0$$

$$\det(sI-A) = (s-1)(-1)^{(3+3)} \det \begin{bmatrix} s-1 & 2 \\ 2 & s-1 \end{bmatrix}$$

$$= (s-1)\left[(s-1)^2 - 4 \right]$$

$(\tau-1)(\tau-1)$

$p(\tau) = 0$ $\qquad (\tau-1)\left[\underbrace{(\tau-1)^2 - 4}_{\tau^2 - 2\tau + 1 - 4} \right] = 0$

$\tau = 1$

$$(\tau-1)(\tau-3)(\tau+1) = 0$$

eigenvalues: $\tau_1 = 1$, $\tau_2 = 3$, $\tau_3 = -1$

Find

A basis for eigenspace of $\tau_1 = 1$

$$\underbrace{A - \tau_1 I}_{M} = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(A - \tau_1 I)\vec{x} = \vec{b} \qquad \vec{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ \cancel{k_4} \end{bmatrix}$$

If $\vec{x} \in \ker(A - \tau_1 I)$ then $\exists$ "there exists" some $k_1, k_2, k_3, \cancel{k_4}$ s.t.

$$\underline{k_1 \vec{v}_1 + k_2 \vec{v}_2 + k_3 \vec{v}_3} + \cancel{k_4 \vec{v}_4} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

equivalent to

$$M\vec{x} = \begin{bmatrix} \underbrace{\begin{matrix} 0 \\ -2 \\ 0 \end{matrix}}_{\vec{v}_1} & \underbrace{\begin{matrix} -2 \\ 0 \\ 0 \end{matrix}}_{\vec{v}_2} & \underbrace{\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}}_{\vec{v}_3} \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{\vec{b}}$$

$$\text{rref}([M, \vec{b}]) = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow \quad x_1 = 0, \; x_2 = 0 \qquad \vec{x} = \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix}$

x_3 : free variable

$$E_{\tau=1} = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a = b = 0 \right\}$$

$\tau_2 = 3$

$M\vec{x} = (A - 3I)\vec{x} = 0$

$$\begin{bmatrix} 1 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} -2 & -2 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$\text{rref}([M, \vec{b}]) = \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 + x_2 = 0$

$x_3 = 0 \qquad \vec{x} = \begin{bmatrix} -x_2 \\ x_2 \\ 0 \end{bmatrix}$

$\Rightarrow x_1 = -x_2$

$$E_{\tau=3} = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a = -b, \; c = 0 \right\}$$

$r_3 = -1$

$$A - (-1)I = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\text{rref}([M, \vec{b}]) = \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 - x_2 = 0 \Rightarrow x_1 = x_2$

$x_3 = 0$

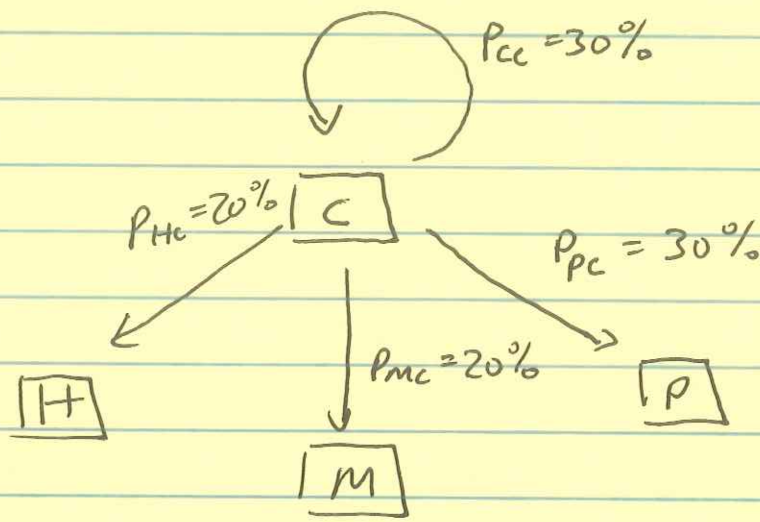
$$\vec{x} = \begin{bmatrix} x_2 \\ x_2 \\ 0 \end{bmatrix}$$

$$E_{r=-1} = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a = b,\ c = 0 \right\}$$

Markovian Process

Consider 4 dinner options:

C	Chinese	
M	Mexican	states
P	Pizza	
H	Home	



We construct a transition matrix

$$A = \begin{array}{c} \\ \\ \\ \\ \end{array} \begin{array}{cccc} H & C & M & P \end{array}$$

$$A = \left[\begin{array}{cccc} .25 & .2 & .25 & .30 \\ .20 & .3 & .25 & .30 \\ .25 & .2 & .40 & .10 \\ .30 & .3 & .10 & .30 \end{array}\right] \begin{array}{c} H \\ C \\ M \\ P \end{array}$$

probability of eating Mexican after eating Chinese!

$[A]_{ij} = a_{ij}$ the probability of state i after I was in state j