Lecture 14

Inner product, norm, orthogonality

Def: $\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$

$$\underbrace{\vec{u} \cdot \vec{v}}_{\text{inner dot product}} = 1\{ [\underbrace{u_1 \dots u_n}_{n}] \underbrace{\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}}_{1} \Big\} n$$

$$= u_1 v_1 + u_2 v_2 + \dots + u_n v_n$$

$$= \vec{u}^T \vec{v} \qquad \vec{v}, \vec{u} \in \mathbb{R}^n$$

$$= (1 \times n)(n \times 1) = \underbrace{(1 \times 1)}_{\text{scalar}}$$

Transpose

Consider matrix A

$$[A]_{ij} = a_{ij}$$

$$B = A^T \Rightarrow [B]_{ij} = a_{ji}$$

Properties of inner product

a) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

b) $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$

c) $(c\vec{u}) \cdot \vec{v} = c(\vec{u} \cdot \vec{v}) = \vec{u} \cdot (c\vec{v})$

d) $\vec{u} \cdot \vec{u} \geq 0$

$\vec{u} \cdot \vec{u} = 0$ if and only if $\vec{u} = 0$

<u>Def</u> (norm)

$$\|\vec{v}\| = \sqrt{\vec{v}\cdot\vec{v}} = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$$

$$\|\vec{v}\|^2 = \vec{v}\cdot\vec{v}$$

<u>Properties of norm</u>

*) $\|\vec{v}\| = 0$ iff $\vec{v} = 0$

*) $\|c\vec{v}\| = |c|\,\|\vec{v}\|$

*) Consider $\vec{v} \neq 0$

Let $\vec{u} = \vec{v}\left(\frac{1}{\|\vec{v}\|}\right)$

$$\Rightarrow \|\vec{u}\| = \left\|\left(\frac{1}{\|\vec{v}\|}\vec{v}\right)\right\| = \left|\frac{1}{\|\vec{v}\|}\right| \|\vec{v}\|$$

$$= \frac{1}{\|\vec{v}\|}\cdot\|\vec{v}\| = 1$$

Ex: $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

<u>Normalize $\vec{v}$:</u>

$$\|\vec{v}\| = \sqrt{(1)^2 + (1)^2 + (2)^2} = \sqrt{6}$$

$$\vec{u} = \frac{1}{\|\vec{v}\|}\cdot\vec{v} = \frac{1}{\sqrt{6}}\cdot\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}$$

<u>Def</u> (distance between vectors)

$dist(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|$

$\underbrace{dist(\vec{u}, \vec{v})}$ distance between $\vec{u}$ and $\vec{v}$

Ex.

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$\vec{u} - \vec{v} = \begin{bmatrix} u_1 - v_1 \\ u_2 - v_2 \\ u_3 - v_3 \end{bmatrix}$$

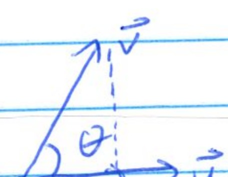
$$dist(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2}$$

<u>Def</u> (orthogonal vectors)

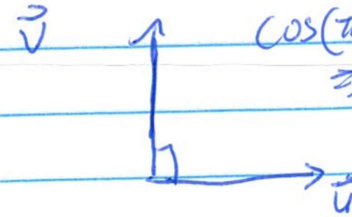
If $\vec{u} \cdot \vec{v} = 0$, then we say $\vec{u}$ and $\vec{v}$ are orthogonal to eachother.

($\vec{u}$ is normal to $\vec{v}$ or $\vec{u}, \vec{v}$ are normal)

In 2D $\quad \vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos(\theta)$

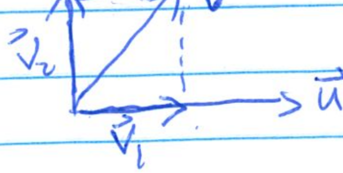


$\theta = 90°$



$\cos(\pi/2) = 0$

$\Rightarrow \vec{v} \cdot \vec{u} = 0$



$\Rightarrow \vec{v} = \vec{v}_1 + \vec{v}_2$

$\vec{v}_1 \cdot \vec{v}_2 = 0$

$$\vec{u}\cdot\vec{u} = \|\vec{u}\|^2\cos\theta \qquad \theta = 0$$
$$= \|\vec{u}\|^2$$

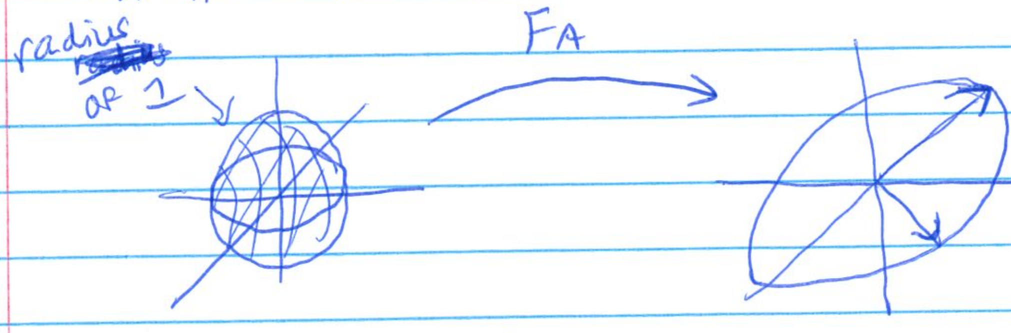
SVD : Singular Value Decomposition

Ex. $A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$ $\quad F_A : \mathbb{R}^3 \to \mathbb{R}^2$

$\hat{A} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ $\quad F_{\hat{A}} : \mathbb{R}^2 \to \mathbb{R}^2$

transformation that stretches a vector by 3 in y-direction and scales by 1 in the x-direction

$F_A : \mathbb{R}^3 \to \mathbb{R}^2$



Decompose or factorize the matrix A as follows

$$A = U \Sigma V^T$$

Σ : rectangular diagonal matrix that contain singular values of A

Examples of Σ

$$\Sigma_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}$$

$$\Sigma_1^+ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \end{bmatrix}$$

U is composed of the eigenvectors of AA^T (left singular vectors of A)
(orthonormal eigenvectors)

V is composed of the eigenvectors of A^TA (right singular vectors of A)
(orthonormal eigenvectors)

Σ: contains the square root of the eigenvalues of AA^T, A^TA

AA^T has eigenvalues τ_1, τ_2 and eigenvectors $\underbrace{\begin{bmatrix} -0.9487 \\ -0.3162 \end{bmatrix}}_{\vec{v}_1}, \underbrace{\begin{bmatrix} 0.3162 \\ -0.9487 \end{bmatrix}}_{\vec{v}_2}$

$\tau_1 = 360 \qquad \tau_2 = 90$

$$U = [\vec{v}_1, \vec{v}_2]$$

$$\Sigma = \begin{bmatrix} \sqrt{360} & 0 & 0 \\ 0 & \sqrt{90} & 0 \end{bmatrix}$$

A^TA

Use similar steps to find V

Applications : Psuedo - inverse

$$A = U \Sigma V^T$$
$$A^+ = V \Sigma^+ U^T$$

Assume

$Ax = b$ $\quad$ $A \in \mathbb{R}^{m \times n}$

$m > n$

and A is consistent

(For square - invertible matrix $x = A^{-1} b$)

solution $\quad x = A^+ b$