Lecture 15

Lecture 14 continued...

(1)

Day 1 Suppose IC (initial condition)

$$X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{array}{l} \leftarrow 100\% \text{ probability } H \\ \leftarrow 0\% \ C \\ \leftarrow 0\% \ M \\ \leftarrow 0\% \ P \end{array}$$

Day 2 What do my dinner options look like?

$$X_2 = A X_1$$

$$= \begin{bmatrix} .25 \\ .20 \\ .25 \\ .30 \end{bmatrix} \begin{array}{l} \leftarrow 25\% \ H \\ \leftarrow 20\% \ C \\ \leftarrow 25\% \ M \\ \leftarrow 30\% \ P \end{array}$$

Day 3

$$X_3 = A \underbrace{X_2}_{\text{day 2 probability distribution}} = A \begin{bmatrix} .25 \\ .20 \\ .25 \\ .30 \end{bmatrix}$$

BUT $X_3 = A X_2 = A(AX_1)$

$$= A^2 X_1 = \begin{bmatrix} .2550 \\ .2625 \\ .2325 \\ .25 \end{bmatrix}$$

Day 5 $X_5 = A^4 X_1 = A^5 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} .2495 \\ .2634 \\ .2339 \\ .2532 \end{bmatrix}$

Day 30?

$$X_{30} = A^{29} X_1 = \begin{bmatrix} .2495 \\ .2634 \\ .2339 \\ .2532 \end{bmatrix}$$

Day 50?

$$X_{50} = A^{49} X_1 = \begin{bmatrix} .2495 \\ .2634 \\ .2339 \\ .2532 \end{bmatrix}$$

Day 30 and Day 50 look the same!

Note: This isn't always the case

What is the state converging to?
Eigenvector with eigenvalue 1

$$Ax = \lambda x$$

$$A^2 x = A(\underbrace{Ax}_{\lambda x}) = \lambda(\underbrace{Ax}_{\lambda x}) = \lambda^2 x$$

$$A^{50} = \lambda^{50} x$$

Can we factorize A? (~~Can we~~ Is A diagonalizable?) Yes if A is an $n \times n$ matrix and has n distinct eigenvalues.

If A is diagonalizable the there exists a matrix P s.t.

$$A = PDP^{-1}$$

$$A^{10} = PD^{10}P^{-1}$$

$$n \to \infty \quad A^n = \begin{bmatrix} .2495 & .2495 & " & " \\ .2634 & .2634 & " & " \\ .2339 & .2339 & " & " \\ .2532 & .2532 & " & " \end{bmatrix}$$

$$x_1 = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} \qquad \sum p_i = 1$$

Complex eigenvalues

$$A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

Step 1: find the eigenvalues

$$sI - A = \begin{bmatrix} s & -2 \\ 2 & s \end{bmatrix}$$

$$\det(sI-A) = s^2+4=0$$
$$s^2 = -4$$
$$s = \pm\sqrt{-4}$$
$$= \pm i\sqrt{4}$$
$$= \pm i2$$

$$\boxed{\tau_1 = i2} \quad \boxed{\tau_2 = -i2}$$

step 2: Find the eigenvectors

$$\underbrace{(A - \tau_1 I)}\, x = 0$$

$$\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} - \begin{bmatrix} i2 & 0 \\ 0 & i2 \end{bmatrix} = \begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix}$$

$$L_1(i) + L_2 \to L_2 \qquad \begin{bmatrix} -2i & 2 \\ 0 & 0 \end{bmatrix}$$

$$-\cancel{2}i\, x_1 + \cancel{2}x_2 = 0$$

$$x_1(-i) = -x_2$$

$$x_1 = -ix_2$$

$$\begin{bmatrix} -ix_2 \\ x_2 \end{bmatrix}$$

$$x_2 \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

$$E_{\tau=2i} = \text{span}\left\{ \begin{bmatrix} -i \\ 1 \end{bmatrix} \right\}$$

Next check $\tau_2 = -i2$

$$A - \tau_2 I = \begin{bmatrix} 2i & 2 \\ -2 & 2i \end{bmatrix}$$

$$L_1(-i) + L_2 \to L_2 \qquad \begin{bmatrix} 2i & 2 \\ 0 & 0 \end{bmatrix}$$

$$E_{\tau=2i} = \text{span}\left\{ \begin{bmatrix} i \\ 1 \end{bmatrix} \right\}$$

$\left\{ \begin{bmatrix} i \\ 1 \end{bmatrix} \right\}$ is a basis for eigenspace of $\tau_2 = -2i$