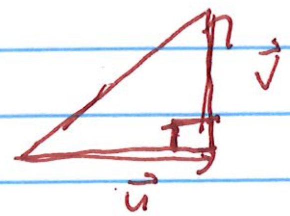
Lecture 15

$$u - v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$$\|u - v\| = \sqrt{(-1)^2 + 0^2 + 0^2} = \sqrt{1} = 1$$

$\vec{u} \perp \vec{v}$ if and only if
$$\|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$$

Pythagorean Theorem



Def: Let W be a subspace of $\mathbb{R}^n$
If $\vec{z} \perp$ to all vectors in W
then we say $\vec{z} \perp W$

$$W = \text{span}\left\{ \overset{\vec{u}}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}, \overset{\vec{v}}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}} \right\} \qquad \vec{z} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{z} \cdot \vec{u} = 0 \qquad \vec{z} \cdot \vec{v} = 0$$

Let $\vec{w} \in W$ then $\vec{w} = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$ $\quad a, b \in \mathbb{R}$

$$\vec{w} \cdot \vec{z} = a \cdot 0 + b \cdot 0 + 0 \cdot 1 = 0$$
$$\vec{z} \perp \vec{w}$$
$$\Rightarrow \vec{z} \perp W$$

Def: (orthogonal complement of a subspace)

$$W^{\perp} = \{ \text{all } \vec{z} \text{ satisfying } \vec{z} \perp W \}$$

In the example above

$$W^{\perp} = \text{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Properties of $W^{\perp}$

*) $W^{\perp}$ is a subspace

*) Suppose $W = \text{span}\{\vec{v}_1, \dots, \vec{v}_p\}$

$\vec{x}$ is in $W^{\perp}$ if and only if

$$\vec{x} \perp \vec{v}_1, \dots, \vec{x} \perp \vec{v}_p$$

Relation between Nulsp A and Rowsp A

Example
$$A = \begin{array}{c} \begin{bmatrix} -1 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix} \begin{array}{l} \} \vec{a}_1 \\ \} \vec{a}_2 \end{array} \\ \begin{array}{ccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{array} \end{array}$$

$$A\vec{x} = 0$$

$$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \cdot x_1 + 0 \cdot x_2 + 2 \cdot x_3 \\ 0 \cdot x_1 + 2 \cdot x_2 + 1 \cdot x_3 \end{bmatrix} = \begin{bmatrix} \vec{a}_1 \cdot \vec{x} \\ \vec{a}_2 \cdot \vec{x} \end{bmatrix}$$

$$\vec{x} \in \text{nullspace } A$$

$$W = \text{span}\{\vec{a}_1, \vec{a}_2\}$$

$$\vec{x} \in W^{\perp}$$

$$\vec{x} \perp \text{rowsp}(A)$$

Conclusion: $\text{Nul } A = (\text{Rowsp } A)^{\perp}$

$\text{Nul } A^T = (\text{Colsp}(A))^{\perp}$

$$A^T = \begin{bmatrix} -1 & 0 \\ 0 & 2 \\ 2 & 1 \end{bmatrix}$$

$A^T x = 0$

$\cancel{\text{colsp}(A) = \text{span}\{a_1, a_2\}}$

$\text{colsp}(A) = \text{span}\{v_1, v_2, v_3\}$

$$\begin{array}{c} \vec{v}_1 \{ \\ \vec{v}_2 \{ \\ \vec{v}_3 \{ \end{array} \begin{bmatrix} -1 & 0 \\ 0 & 2 \\ 2 & 1 \end{bmatrix} \vec{x} = 0 \qquad \vec{x} \in \mathbb{R}^2$$

Dimension of W and $W^{\perp}$

$W =$ a subspace of $\mathbb{R}^n$

Let $\{\vec{v}_1, \dots, \vec{v}_p\}$ be a basis for W

$\Rightarrow W = \text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\}$

Let $A = [\vec{v}_1, \dots, \vec{v}_p]$ $A \in \mathbb{R}^{n \times p}$

$W = \text{Colsp } A$

$\dim W = \dim(\text{colsp}(A)) = \text{rank } A = \text{rank } A^T$

$\dim W^{\perp} = \dim(\text{Colsp}(A)^{\perp}) = \dim(\text{Nul } A^T)$

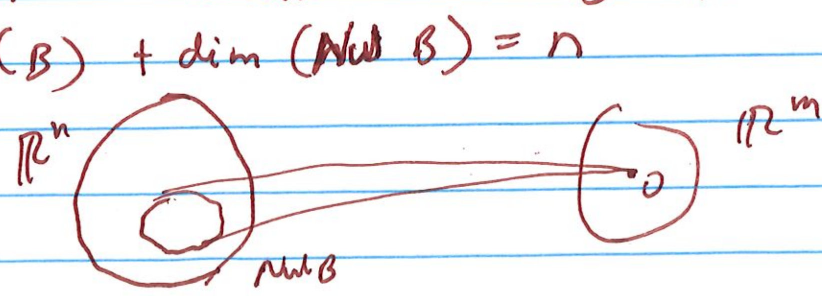
Nul

Recall rank-nullity theorem

for $B \in \mathbb{R}^{m \times n}$ $F_B : \mathbb{R}^n \to \mathbb{R}^m$

$\text{rank}(B) + \dim(\text{Nul } B) = n$

$\text{Nul } B \equiv \text{Ker}(B)$



Apply it to A^T

$$\operatorname{rank}(A^T) + \dim(\operatorname{Nul} A^T) = n$$

$$\Downarrow$$

$$\dim(W) + \dim(W^\perp) = n$$

TH 4

$$\operatorname{colsp} A = \operatorname{Span}\left\{ \underset{\vec{v}_1}{\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}}, \underset{\vec{v}_2}{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}} \right\} \qquad v_1 \in \mathbb{R}^3$$

Is $\operatorname{colsp} A = \mathbb{R}^3$? No

requires $\dim(\operatorname{colsp}(A)) = 3$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 2 & -1 \end{bmatrix} \qquad \dim(\operatorname{colsp}(A)) = 2$$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 2 & 2 \end{bmatrix} \qquad \dim(\operatorname{colsp}(A)) = 1$$

Applied rank-nullity theorem

$$\dim(W) + \dim(W^\perp) = \dim(\mathbb{R}^3)$$

$$2 + ? = 3$$

$$A^T x = 0 \Rightarrow x \in W^\perp$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 1 & 1 & -1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 0 & -3 & 0 \end{array}\right]$$

$(-1)L_1 + L_2 \to L_2$

$$x_1 + x_2 + 2x_3 = 0 \Rightarrow x_1 = -x_2$$
$$-3x_3 = 0 \Rightarrow x_3 = 0$$

All solutions are span $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\} = W^\perp$

$\dim(W^\perp) = 1$

Question on HW:

- Find the orthogonal complement of W
- Find $\dim(W^\perp)$
- Find a basis for $W^\perp$

Orthogonal Set

Def : $\{\vec{u}_1, \dots, \vec{u}_p\}$ is called an orthogonal set if

$$\vec{u}_i \cdot \vec{u}_j = 0 \quad \text{for } i \neq j$$

An orthogonal set of non-zero vectors is linearly independent

Def : If a basis for W is an orthogonal set then it is called an orthogonal basis for W

Ex : $\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $\vec{u}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

show that $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthogonal basis for $\mathbb{R}^3$

1) check that they are orthogonal

$$\left.\begin{aligned} \vec{u}_1 \cdot \vec{u}_2 &= 0 \\ \vec{u}_1 \cdot \vec{u}_3 &= 0 \\ \vec{u}_2 \cdot \vec{u}_3 &= 0 \end{aligned}\right\} \text{ If true then set is linearly independent}$$

2) Show $\text{span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3\} = \mathbb{R}^3$

3 linearly independent vectors in $\mathbb{R}^3$ span $\mathbb{R}^3$

$$\boxed{\text{An orthogonal set of } n \text{ non-zero vectors in } \mathbb{R}^n \text{ is an orthogonal basis for } \mathbb{R}^n}$$

<u>Theorem</u> Let $\{\vec{u}_1, \ldots, \vec{u}_p\}$ be an orthogonal basis for W

Each $\vec{y} \in W$ can be written as a linear combination of basis vectors

$$\vec{y} = c_1\vec{u}_1 + .. + c_p\vec{u}_p$$

$$\left(\vec{y} = c_1\vec{u}_1 + .. + c_p\vec{u}_p\right) \cdot \vec{u}_2$$

remaining term

$$\vec{y} \cdot \vec{u}_2 = c_1 \cancel{\vec{u}_1 \cdot \vec{u}_2}^{\,0} + c_2 \vec{u}_2 \cdot \vec{u}_2 + \ldots + c_p \cancel{\vec{u}_p \cdot \vec{u}_2}^{\,0}$$

$$\vec{y} \cdot \vec{u}_2 = c_2 \vec{u}_2 \cdot \vec{u}_2$$

$$\Rightarrow \quad c_2 = \frac{\vec{y} \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2}$$

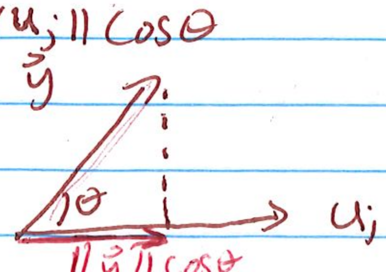
The coefficients $\{c_j\}$ are given by

$$\boxed{c_j = \frac{\vec{y} \cdot \vec{u}_j}{\underbrace{\vec{u}_j \cdot \vec{u}_j}_{\substack{\|u\|^2 \\ \text{normalization} \\ \text{factor}}}}}$$

← projection of $\vec{y}$ onto span $\{\vec{u}_j\}$

2D example

$$\vec{y} \cdot \vec{u}_j = \|y\| \|u_j\| \cos\theta$$



8

Ex

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \vec{y} = \begin{bmatrix} 6 \\ 2 \\ 5 \end{bmatrix}$$

$$\vec{y} = c_1 \vec{u}_1 + c_2 \vec{u}_2 + c_3 \vec{u}_3$$

$$c_1 = \frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} = \frac{6 + 2 + 10}{1 + 1 + 4} = \frac{18}{6} = 3$$

$$c_2 = \frac{\vec{y} \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2} = \frac{6 + 2 - 5}{1 + 1 + 1} = \frac{3}{3} = 1$$

$$c_3 = \frac{\vec{y} \cdot \vec{u}_3}{\vec{u}_3 \cdot \vec{u}_3} = \frac{-6 + 2 + 0}{1 + 1 + 0} = \frac{-4}{2} = -2$$

$$(\vec{y} = c_1 \vec{u}_1 + c_2 \vec{u}_2 + c_3 \vec{u}_3) \cdot \vec{u}_2$$

$$\underline{\underline{\vec{y} \cdot \vec{u}_2}} = c_1 \underbrace{\vec{u}_1 \cdot \vec{u}_2}_{=0} + \underline{\underline{c_2 \vec{u}_2 \cdot \vec{u}_2}} + c_3 \underbrace{\vec{u}_3 \cdot \vec{u}_2}_{=0}$$

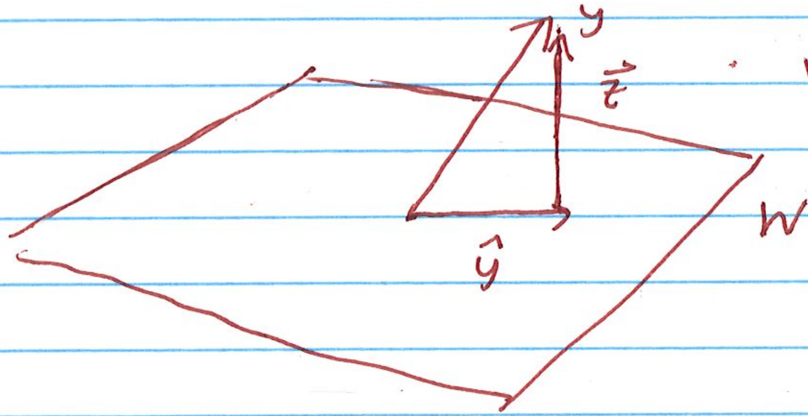
$$c_2 = \frac{\vec{y} \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2}$$

Orthogonal Projection

Theorem (orthogonal decomposition theorem)

Let W be a subspace of $\mathbb{R}^n$

Let $\{\vec{u}_1, \dots, \vec{u}_p\}$ be an orthogonal basis for W



$W + W^\perp = \mathbb{R}^n$

$\hat{y} \in W$

Each $\vec{y}$ in $\mathbb{R}^n$ can be decomposed as $\vec{y} = \hat{y} + \vec{z}$

where $\hat{y}$ is in W and $\vec{z}$ is in $W^\perp$

one solution for $\hat{y}$ is given by

$$\hat{y} = \left(\frac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\right)\vec{u}_1 + \dots + \left(\frac{\vec{y}\cdot\vec{u}_p}{\vec{u}_p\cdot\vec{u}_p}\right)\vec{u}_p$$

$\hat{y}$ is unique (projection of $\vec{y}$ onto W is unique)

$\hat{y}$ is called the orthogonal projection of $\vec{y}$ onto W

Notation $\text{proj}_W \vec{y} = \hat{y}$

Theorem (Best representation theorem)

Let W be a subspace of $\mathbb{R}^n$

Let $\vec{y}$ be a vector in $\mathbb{R}^n$

We have

$$\|\vec{y} - \text{proj}_W \vec{y}\| < \|\vec{y} - \vec{v}\|$$

for all $\vec{v} \in W$ and

$\vec{v} \neq \text{proj}_W \vec{y}$ in W