HW #7

Problem 1

a. Use Matlab function eig

b. Find the row canonical form of A

Find the eigenvalues of reduced matrix

c.

$$A^T = \begin{bmatrix} -1 & 0 & 2 \\ 2 & 5 & -1 \\ 4 & 2 & 9 \end{bmatrix}$$

Problem 2

a. Sufficient condition would n distinct eigenvalues

Necessary and sufficient check for linearly independent eigenvectors

b. Do a.

c. $P = [\vec{v}_1, \vec{v}_2]$ constructed from eigenvectors

d. $A = PDP^{-1} \rightarrow D = P^{-1}AP$

scalar example $a = p d p^{-1}$

$a = \frac{p}{p} d = d$

OR $p^{-1}(a = pdp^{-1})\,p$

$p^{-1}ap = pdp^{-1}$ for non-scalar

e.

e. Find P^{-1}

Inverse of a 2x2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

f. $$(PD\underbrace{P^{-1})(P}_{I}DP^{-1}) = (PDP^{-1})^2$$

$$PDDP^{-1} = PD^2P^{-1}$$

g. Useful definition

A and B are called similar if there exists P such that

$$B = P^{-1}AP$$

Theorem A and B are similar

$\Rightarrow$ They have the same characterstic polynomial (same set of eigenvalues and algebraic mutiplicities)

They also have the same geometric mutiplicities)

$Nul(A - \lambda I) \longleftrightarrow Nul(B - \lambda I)$

one-to-one correspondence

$$\underset{M}{\underset{\sim}{A}}^3 = PD^3P^{-1}$$

$$A = \begin{bmatrix} 1 & -1 \\ & 2 \end{bmatrix} \qquad \tau_1 = 1, \ \tau_2 = 2$$

I want to know eigenvalues of A^5

1st find eigenvalues of A, check that it is diagonalizable

$\exists$ a P s.t.

$$A = PDP^{-1}$$

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$A^5 = PD^5P^{-1} = P\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^5 P^{-1}$$

$$A^5 = P\begin{bmatrix} 1^5 & 0 \\ 0 & 2^5 \end{bmatrix} P^{-1}$$

A^5 has eigenvalues $\tau_1 = 1$ $\tau_2 = 2^5$

Problem 3

a) $p(s) = \det(sI - A)$

b) Example:

Suppose

$\det(sI - A) = s^2 + bs + c$ $\quad A = \begin{bmatrix} 1 & 0 \\ -1 & 5 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 \\ -1 & 5 \end{bmatrix}^2 + b\begin{bmatrix} 1 & 0 \\ -1 & 5 \end{bmatrix} + \underbrace{\begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}}_{cI} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

c,d Find eigenvalues A, $A - \tau_i I$

e. show $(A - \tau_i I) v_i = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Show $A v_i = \tau_i v_i$

Problem 4

look for complex eigenvalues and eigenvectors

Reference Thursday's notes

Problem 5

$$A^T = \begin{bmatrix} -5 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 3 & 2 & 2 & 3 \end{bmatrix}$$

~~p(s) =~~ Eigenvalues of A^T equals

" of A

$p(s) = \det(sI - A)$

$$\det(A^T) = -5(-1)^2 \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 2 & 3 \end{bmatrix}$$

$$= (-5)(-1)^2\, 1\, (-1)^2 \det \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$$

$$= (-5)(-1)^2\, 1\, (-1)^2 (3)$$

Example

Case 1 $\quad A = \begin{bmatrix} 1 & 5 & 3 & 1 \\ 0 & 2 & 4 & 5 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 7 \end{bmatrix}$

eigenvalues $\lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 3,\ \lambda_4 = 7$

4 distinct eigenvalues $\Rightarrow$ A is diagonalizable

Case 2

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 4 \\ 0 & 0 & 2 \end{bmatrix} \qquad \begin{bmatrix} s-2 & 0 & 0 \\ -1 & s-3 & -4 \\ 0 & 0 & s-2 \end{bmatrix}$$

$$\det(sI - A) = \cancel{2(-1)^2 \det}$$

$$= (s-2)(-1)^2 (s-3)(s-2)$$

$$= (s-2)^2 (s-3)$$

$\tau_1 = 2$ ← algebraic multiplicity of 2
$\tau_2 = 2$
$\tau_3 = 3$

Find a basis for $\tau = 2$

$$A - 2I = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[\text{rref}]{} \begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Soltn

$x_1 = -x_2 - 4x_3$ 2 free variables

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 - 4x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{basis} = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$\tau = 2$ has geometric multiplicity of 2

geometric multiplicity equals my algebraic multiplicity

$\Rightarrow A$ is diagonalizable

Case 3

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 4 \\ 1 & 0 & 2 \end{bmatrix} \Rightarrow \begin{array}{l} \lambda_1 = 2 \\ \lambda_2 = 2 \\ \lambda_3 = 3 \end{array}$$

$\lambda = 2$

$$A - \lambda I = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 4 \\ 1 & 0 & 0 \end{bmatrix} \xrightarrow[\text{rref}]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

1 Free variable

$$x_1 = 0$$
$$x_2 = -4x_3$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -4x_3 \\ x_3 \end{bmatrix} = x_3 \underbrace{\begin{bmatrix} 0 \\ -4 \\ 1 \end{bmatrix}}_{\text{basis for } \lambda = 2 \text{ eigenspace}}$$

geometric multiplicity one

$\Rightarrow$ A is <u>not</u> diagonalizable

Inner product, norm, orthogonality

<u>Def</u> : $\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$

$$\underbrace{\vec{u} \cdot \vec{v}}_{\substack{\text{Inner product} \\ \text{dot product}}} \equiv 1\{\underbrace{[u_1 \; \dots \; u_n]}_{n} \underbrace{\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}}_{1} \Big\} n$$

$$= u_1 v_1 + u_2 v_2 + \dots + u_n v_n$$

$$= \vec{u}^T \vec{v} \qquad \vec{v}, \vec{u} \in \mathbb{R}^n$$

$$(1 \times n)(n \times 1) = \underset{\text{Scalar}}{1 \times 1}$$

Properties of inner product

a) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

b) $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$

c) $(c\vec{u}) \cdot \vec{v} = c(\vec{u} \cdot \vec{v}) = \vec{u} \cdot (c\vec{v})$

d) $\vec{u} \cdot \vec{u} \geq 0$

$\vec{u} \cdot \vec{u} = 0$ if and only if $\vec{u} = 0$

<u>Def</u> (norm)

$$\|\vec{v}\| \equiv \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$$

$$\|\vec{v}\|^2 = \vec{v} \cdot \vec{v}$$

Properties of norm

*) $\|\vec{v}\| = 0$ if and only if $\vec{v} = 0$

*) $\|c\vec{v}\| = |c|\,\|\vec{v}\|$

*) Consider $\vec{v} \neq 0$

Let $\vec{u} = \vec{v}\left(\frac{1}{\|\vec{v}\|}\right)$

$$\Rightarrow \|\vec{u}\| = \left\| \left(\underbrace{\frac{1}{\|\vec{v}\|}}_{\text{constant}} \vec{v} \right) \right\| = \left| \frac{1}{\|\vec{v}\|} \right| \|\vec{v}\| = 1$$

Ex: $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

Normalize $\vec{v}$:

$$\|\vec{v}\| = \sqrt{(1)^2 + (1)^2 + (2)^2} = \sqrt{6}$$

$$\vec{u} = \frac{1}{\|\vec{v}\|} \vec{v} = \frac{1}{\sqrt{6}} \cdot \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}$$

Def. (distance between two vectors)

$$\underbrace{\text{dist}(\vec{u}, \vec{v})}_{\text{distance between } \vec{u} \text{ and } \vec{v}} \equiv \|\vec{u} - \vec{v}\|$$

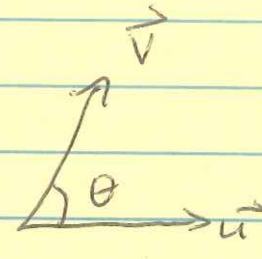
Ex. $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

$$\vec{u} - \vec{v} = \begin{bmatrix} u_1 - v_1 \\ u_2 - v_2 \\ u_3 - v_3 \end{bmatrix}$$

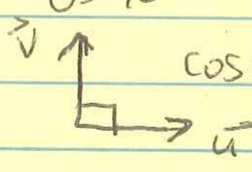
$$\text{dist}(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2}$$

Def (orthogonal vectors)
If $\vec{u} \cdot \vec{v} = 0$, then we say $\vec{u}$ and $\vec{v}$ are orthogonal to eachother

In 2D $\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos(\theta)$



$\theta = 90°$



$\cos(\pi/2) = 0$

$$\vec{u} \cdot \vec{u} = \|\vec{u}\|^2 \cos(\theta) \qquad \theta = 0$$
$$= \|\vec{u}\|^2$$