Recap

Last lecture we looked at examples of quasilinear systems of the form
\[ a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u) \]
and solved using the method of characteristics.

\[ \frac{dx}{ds} = 0 \quad \frac{dy}{ds} = 5 \quad \frac{du}{ds} = 2 \]

Then we returned to wave equation using one of C we recovered d’Alambert’s solution.

So far we have assumed infinite spatial domain.

Today: we discuss wave reflections.

Start with some definitions:
\[ x - ct = \text{const} \]
\[ x + ct = \text{const} \]

What are the characteristics that intersect \((x_0, t_0)\)?
Solution at this point \((x_0, t_0)\)

\[
U(x_0, t_0) = \frac{f(x_0 + ct_0) + f(x_0 - ct_0)}{2} + \frac{1}{2c} \int_{x_0 - ct_0}^{x_0 + ct_0} g(z) \, dz
\]

\(U(x_0, t_0)\) is function of \(f\) at the base points of triangle \(x_0 - ct_0\) and \(x_0 + ct_0\) and \(g\) along the base.

That is \(U(x_0, t_0)\) depends only on initial data given on the interval \([x_0 - ct_0, x_0 + ct_0]\).

In particular, if on this interval \(g\) are smooth, then the solution \(U\) stays smooth in the characteristic triangle.
Say we ask the opposite question:

What points are influenced by a set of ICs?

Region of influence of the interval \([a, b]\)

IC specified on \([a, b]\)

Back to the wave equation

Wave equation on a half-line:

\[
\begin{align*}
\frac{\partial u}{\partial t} + c^2 \frac{\partial^2 u}{\partial x^2} &= 0 & x > 0, t > 0 \\
U(x, 0) &= f(x) & x > 0 \\
U(x, t) &= g(x) & x > 0 \\
U(0, t) &= 0 & t > 0
\end{align*}
\]

1st extend \(f\) and \(g\) to all \(\mathbb{R}\) by odd reflection.

Do same for \(g(x)\)
Mathematically speaking

\[ f(x) = \begin{cases} \frac{1}{2} f(x) & x > 0 \\ -f(-x) & x \leq 0 \end{cases} \]
\[ g(x) = \begin{cases} \frac{1}{2} g(x) & x > 0 \\ -g(-x) & x \leq 0 \end{cases} \]

Now we can apply d'Alambert's Solution

We know d'Alambert's solution solves

\[ \tilde{u}_{tt} - c^2 \tilde{u}_{xx} = 0 \quad t > 0 \]
\[ \tilde{u}(x,0) = \tilde{f}(x) \quad -\infty < x < \infty \]
\[ \tilde{u}_t(x,0) = \tilde{g}(x) \]

Note: \( u(x,t) = \tilde{u}(x,t) \) on \( x \in [0,\infty) \)

Similarly

\[ u(x,0) = \tilde{u}(x,0) \quad x \in [0,\infty) \]
\[ u_t(x,0) = \tilde{u}_t(x,0) \]

It remains to show is \( u(0,t) = 0 \)

\( \Rightarrow \) suffices to show that \( \tilde{u}(x,t) \) is an odd function for \( t > 0 \)
Let \( v(x,t) = -\tilde{u}(-x,t) \)

\[
\Rightarrow \quad v_x(x,t) = \tilde{u}_x(-x,t) \quad v_t(x,t) = -\tilde{u}_t(-x,t)
\]

\[
v_{xx}(x,t) = -\tilde{u}_{xx}(-x,t) \quad v_{tt}(x,t) = -\tilde{u}_{tt}(-X,t)
\]

\[
v_{tt} - c^2 v_{xx} = -\tilde{u}_{tt}(-x,t) + c^2 \tilde{u}_{xx}(-x,t)
\]

\[
= -\left( \tilde{u}_{tt}(-x,t) - c^2 \tilde{u}_{xx}(-x,t) \right) = 0
\]

\[
\Rightarrow \quad v_{tt} - c^2 v_{xx} = 0
\]

Look IC's

\[
v(t,0) = -\tilde{u}(-x,0) = -\tilde{f}(-x) = \tilde{f}(x)
\]

\[
v_t(t,0) = -\tilde{u}_t(-x,0) = -\tilde{g}(-x) = \tilde{g}(x)
\]

Solution \( v(x,t) \) with IC. \( v(t,0) = \tilde{f}(x) \quad v_t(t,0) = \tilde{g}(x) \)

is \( \tilde{u}(x,t) \)

\[
v(x,t) = -\tilde{u}(x,t) = \tilde{u}(x,t)
\]

We have shown \( \tilde{u}(x,t) \) is an odd function

\[
\Rightarrow \quad u(0,t) = 0
\]
Look at characteristic curves.

$X = X + Ct$

$X = Ct$

Region of influence of the "wall" of "fixed" point

$0 = X + Ct$

$0 = X - Ct$

If we are looking in domain $X > Ct$

$\Rightarrow$ d'Alambert's solution holds here

Example: Assume $y(t) = 0$ no initial velocity

$(\frac{dy}{dt}(x,0) = 0)$

Reflected wave takes on solution trajectory emanating symmetrically from negative $x$-axis
Example 2

Reflection of shallow water waves from a cliff

Before (with fixed end)

Now we try an even extension of IC's to the left $-\infty < x < 0$

\[ \frac{2u}{\partial x} = 0, \quad x = 0, \quad t > 0 \]

\[ u(x, 0) = f(x), \quad u_t (x, 0) = g(x) \]

\[ \tilde{f}(x) = \begin{cases} f(x) & x > 0 \\ f(-x) & x < 0 \end{cases} \quad \tilde{g}(x) = \begin{cases} g(x) & x > 0 \\ g(-x) & x < 0 \end{cases} \]

$x > ct \iff$ d'Alambert's solution

What about for $0 < x < ct$
For $0 < x < ct$

\[ \tilde{u} = \frac{1}{2} \left[ f(x+ct) + f(x-ct) \right] + \frac{1}{2c} \int_{x-ct}^{x+ct} \tilde{g}(\tau) \, d\tau \]

\[ = \frac{1}{2} \left[ f(x+ct) + f(ct-x) \right] + \frac{1}{2c} \left[ \int_{0}^{x+ct} \tilde{g}(\tau) \, d\tau - \int_{0}^{x-ct} \tilde{g}(\tau) \, d\tau \right] \]

\[ + \frac{1}{2c} \left[ \left( \int_{0}^{ct} - \int_{0}^{ct} \right) \tilde{g}(\tau) \, d\tau \right] - \int_{0}^{ct-x} \tilde{g}(\tau) \, d\tau - \int_{0}^{ct+x} \tilde{g}(\tau) \, d\tau \]

\[ = \frac{1}{2} \left[ f(x+ct) + f(ct-x) \right] + \frac{1}{2c} \left( \frac{ct-x}{ct-x} + \int_{ct-x}^{ct+x} \tilde{g}(\tau) \, d\tau \right) \]

\[ \text{Summarized and continued next lecture} \]