Pythagorean Theorem  
\[ a^2 + b^2 = c^2 \]
\[ c = \sqrt{a^2 + b^2} \]

\[ \langle \vec{u}, \vec{v} \rangle \text{ if and only if } \]
\[ ||\vec{u} + \vec{v}||^2 = ||\vec{u}||^2 + ||\vec{v}||^2 \]

**Def.** Let \( W \) be a subspace of \( IR^n \). If \( \vec{z} \bot \) all vectors in \( W \), then we say \( \vec{z} \bot W \).

Consider subspace \( W = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\} \)

If \( \vec{z} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \) then \( \vec{z} \bot W \)

\[ \vec{z} \cdot \vec{u} = 0, \quad \vec{z} \cdot \vec{v} = 0 \]

Let \( \vec{\omega} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \)

\[ \vec{\omega} \cdot \vec{z} = a \cdot 0 + b \cdot 0 + 0 \cdot 1 = 0 \]

\[ \vec{z} \bot \vec{\omega} \]

**Def.** (orthogonal complement of a subspace)
\[ W^\perp = \left\{ \vec{z} \text{ satisfying } \vec{z} \bot W \right\} \]

In the example \( W^\perp = \text{span} \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \)
Properties of $W^\perp$

1) $W^\perp$ is a subspace

2) Suppose $W = \text{span}\left\{ \vec{v}_1, \ldots, \vec{v}_p \right\}$

$\vec{x}$ is in $W^\perp$ if and only if

$\vec{x} \perp \vec{v}_1, \ldots, \vec{v}_p$

$\vec{x} \in \text{ker}(A) / \text{nullsp}(A) / \text{null space of } A$

Relation between $\text{Nul}A$ and $\text{Rowsp}A$

Example: $A = \begin{bmatrix} -1 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix}$

So when we solve

$AX = 0$

$\begin{bmatrix} \vec{a}_1 \cdot x \\ \vec{a}_2 \cdot x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x \perp \vec{a}_1$

$x \perp \vec{a}_2$

Solution basis for $x$

$\vec{a}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 2 & -\frac{1}{2} & 1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$(1 \times 3) (3 \times 1) = (1 \times 1)$

$\Rightarrow \vec{x} \perp (\text{Row}A)$

Also, $x \in \text{Nul}A$
Conclusion: \( \text{Nul} A = (\text{Row} A)^\perp \)

\[ \Rightarrow \text{Nul} A^T = (\text{Col} A)^\perp \]

\( \Rightarrow \) When we take the transpose, rows become the columns.

Dimensions of \( W \) and \( W^\perp \):

\( W = \) a subspace of \( \mathbb{R}^n \)

let \( \{ \bar{v}_1, ..., \bar{v}_p \} \) be a basis for \( W \)

\[ \Rightarrow W = \text{span} \{ \bar{v}_1, ..., \bar{v}_p \} \]

let \( A = \begin{bmatrix} \bar{v}_1 & \cdots & \bar{v}_p \end{bmatrix} A \in \mathbb{R}^{n \times p} \)

\[ W = \text{colsp} A \]

\( \dim W = \dim (\text{colsp} A) = \text{rank} A = \text{rank} A^T \)

\( \dim W^\perp = \dim (\text{colsp} A)^\perp = \dim \text{Nul} A^T \)

Recall, rank-nullity theorem:

for \( B \in \mathbb{R}^{m \times n} \), \( F^B : \mathbb{R}^n \to \mathbb{R}^m \)

\[ \text{rank}(B) + \dim(\text{Nul} B) = n \]

Apply it to \( A^T \) \( A^T \in \mathbb{R}^{p \times n} \)

\[ \text{rank}(A^T) + \dim(\text{Nul} A^T) = n \]

\[ \downarrow \]

\( \dim(W) + \dim(W^\perp) = n \)
Ex

\[
A = \begin{bmatrix}
1 & 1 \\
1 & 1 \\
2 & -1
\end{bmatrix}
\]

a) Find \( \dim (\text{Colsp} A) \)

b) Find a basis for \( (\text{Colsp} A)^\perp \)

a) \( \text{Colsp} A \) is a subspace in \( \mathbb{R}^3 \)

\[ \dim (\text{Colsp} A) = 2 \]

\[ \Rightarrow \dim (\text{Colsp} A)^\perp = 3 - \dim (\text{Colsp} A) = 1 \]

b) \( A^T = \begin{bmatrix}
1 & 1 & 2 \\
1 & 1 & -1
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \]

pivot variable \( x_1, x_3 \)

free variable \( x_2 \)

Solution \( A^T x = 0 \)

\[ x_1 + x_2 = 0 \]

\[ x_3 = 0 \]

\[ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \]

\[ \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\} \text{ is a basis for } \text{Nul} A^T = (\text{Colsp} A)^\perp \]
Orthogonal Set

Def: \( \{ \vec{u}_1, \ldots, \vec{u}_p \} \) is called an orthogonal set if
\[ \vec{u}_i \cdot \vec{u}_j = 0 \text{ for } i \neq j \]

An orthogonal set of non-zero vectors is linearly independent.

Def: If a basis for \( W \) is an orthogonal set then it is called an orthogonal basis for \( W \).

Ex. \( \vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \vec{u}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \)

Show that \( \{ \vec{u}_1, \vec{u}_2, \vec{u}_3 \} \) is an orthogonal basis for \( \mathbb{R}^3 \).

1) Check that they are orthogonal
\[ \vec{u}_1 \cdot \vec{u}_2 = 1 + 1 - 2 = 0 \]
\[ \vec{u}_1 \cdot \vec{u}_3 = -1 + 1 + 0 = 0 \]
\[ \vec{u}_2 \cdot \vec{u}_3 = -1 + 1 + 0 = 0 \]

\( \Rightarrow \) \( \{ \vec{u}_1, \vec{u}_2, \vec{u}_3 \} \) is an orthogonal set
\( \Rightarrow \) linearly independent
2) Show \( \text{span } \{ \vec{u}_1, \vec{u}_2, \vec{u}_3 \} = \mathbb{R}^3 \)

4 vectors \( = 3 \)
\( \dim \mathbb{R}^3 = 3 \)

and since the 3 vectors are linearly independent they span \( \mathbb{R}^3 \)
\( \Rightarrow \{ \vec{u}_1, \vec{u}_2, \vec{u}_3 \} \) is an orthogonal basis for \( \mathbb{R}^3 \)

An orthogonal set of \( n \) non-zero vectors in \( \mathbb{R}^n \) is an orthogonal basis for \( \mathbb{R}^n \)

Theorem Let \( \{ \vec{u}_1, \ldots, \vec{u}_n \} \) be an orthogonal basis for \( W \)

Each \( \vec{y} \) in \( W \) can be represented as
\( \vec{y} = c_1 \vec{u}_1 + \ldots + c_n \vec{u}_n \)

The coefficients \( c_i \) are given by
\[
 c_i = \frac{\vec{y} \cdot \vec{u}_i}{\vec{u}_i \cdot \vec{u}_i}
\]

Recall in 2D
\[
 \vec{y} \cdot \vec{u}_i = ||y|| ||u_i|| \cos(\theta)
\]

\[
 \frac{\vec{y} \cdot \vec{u}_i}{||u_i||^2}
\]

normalization factor

\[
 \frac{1}{||y|| \cos \theta}
\]
$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$

$\mathbf{y} = c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 + c_3 \mathbf{u}_3$

$c_1 = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} = \frac{6 + 2 + 10}{1 + 1 + 1} = 3$

$c_2 = \frac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2} = \frac{6 + 2 - 5}{1 + 1 + 1} = \frac{3}{3} = 1$

$c_3 = \frac{\mathbf{y} \cdot \mathbf{u}_3}{\mathbf{u}_3 \cdot \mathbf{u}_3} = \frac{-6 + 2 + 0}{1 + 1 + 0} = \frac{-4}{2} = -2$

$(\mathbf{y} = c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 + c_3 \mathbf{u}_3) \cdot \mathbf{u}_2$

$\mathbf{y} \cdot \mathbf{u}_2 = c_1 \mathbf{u}_1 \cdot \mathbf{u}_2 + c_2 \mathbf{u}_2 \cdot \mathbf{u}_2 + c_3 \mathbf{u}_3 \cdot \mathbf{u}_2$

$= 0$

$c_2 = \frac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}$
Orthogonal Projection

Theorem

(orthogonal decomposition theorem)

Let \( W \) be a subspace of \( \mathbb{R}^n \).
Let \( \tilde{y}, \tilde{u}_1, \ldots, \tilde{u}_p \) be an orthogonal basis for \( W \).

Each \( \tilde{y} \) in \( \mathbb{R}^n \) can be decomposed as
\[
\tilde{y} = \tilde{y}_W + \tilde{z}
\]
where \( \tilde{y}_W \) is in \( W \) and \( \tilde{z} \) is in \( W^\perp \).

One solution for \( \tilde{y}_W \) is given by
\[
\tilde{y}_W = \left( \frac{\tilde{y} \cdot \tilde{u}_1}{\tilde{u}_1 \cdot \tilde{u}_1} \right) \tilde{u}_1 + \cdots + \left( \frac{\tilde{y} \cdot \tilde{u}_p}{\tilde{u}_p \cdot \tilde{u}_p} \right) \tilde{u}_p
\]

\( \tilde{y}_W \) is unique

\( \tilde{y} \) is called the orthogonal projection of \( \tilde{y} \) onto \( W \).

Notation: \( \text{proj}_W \tilde{y} \)
Theorem (Best Representation Theorem)
Let $W$ be a subspace of $\mathbb{R}^n$
Let $\vec{y}$ be a vector in $\mathbb{R}^n$

We have
$$\|\vec{y} - \text{proj}_W \vec{y}\| \leq \|\vec{y} - \vec{v}\|$$

for all $\vec{v} \in W$ and
$\vec{v} \neq \text{proj}_W \vec{y}$ in $W$