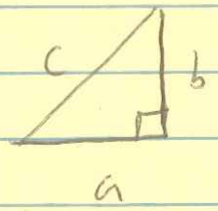
Pythagorean Theorem



$$a^2 + b^2 = c^2$$
$$c = \sqrt{a^2+b^2}$$

$\vec{u} \perp \vec{v}$ if and only if

$$\boxed{\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2}$$

Def: Let W be a subspace of $\mathbb{R}^n$

If $\vec{z} \perp$ all vectors in W

then we say $\vec{z} \perp W$

Consider subspace

$$W = \operatorname{span}\left\{ \underbrace{\begin{bmatrix}1\\0\\0\end{bmatrix}}_{\vec{u}}, \underbrace{\begin{bmatrix}0\\1\\0\end{bmatrix}}_{\vec{v}} \right\}$$

if $\vec{z} = \begin{bmatrix}0\\0\\1\end{bmatrix}$

then $\vec{z} \perp W$

$$\vec{z}\cdot\vec{u} = 0, \quad \vec{z}\cdot\vec{v} = 0$$

Let $\vec{w} = \begin{bmatrix}a\\b\\0\end{bmatrix}$ $\qquad \vec{w}\cdot\vec{z} = a\cdot 0 + b\cdot 0 + 0\cdot 1 = 0$

$$\vec{z} \perp \vec{w}$$

Def: (orthogonal complement of a subspace)

$$W^\perp = \{ \text{all } \vec{z} \text{ satisfying } \vec{z} \perp W \}$$

In the example $\quad W^\perp = \operatorname{span}\left\{ \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$

Properties of $W^\perp$

*) $W^\perp$ is a subspace

*) Suppose $W = \text{span}\{\vec{v}_1, \dots, \vec{v}_p\}$

$\vec{x}$ is in $W^\perp$ if and only if
$\vec{x} \perp \vec{v}_1, \dots, \vec{x} \perp \vec{v}_p$

Ker(A) / Nulsp(A) / null space of A

Relation between NulspA and RowspA

Example
$$A = \begin{bmatrix} -1 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix} \begin{matrix} \} \vec{a}_1 \\ \} \vec{a}_2 \end{matrix}$$

So when we solve

$$Ax = 0$$

$$\begin{bmatrix} \vec{a}_1 \cdot x \\ \vec{a}_2 \cdot x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} x \perp \vec{a}_1 \\ x \perp \vec{a}_2 \end{matrix}$$

Solution basis for x

$$\left\{ \begin{bmatrix} 2 \\ -1/2 \\ 1 \end{bmatrix} \right\}$$

$$x \cdot \vec{a}_1 = x^T \vec{a}_1$$

$$\begin{bmatrix} 2 & -1/2 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} = -2+0+2 = 0$$

$$(1\times 3)\ (3\times) = (1\times 1)$$

$$\Rightarrow \vec{x} \perp (\text{Row} A)$$

~~but~~ Also, $x \in \text{Nul} A$

Conclusion: $\mathrm{Nul}\, A = (\mathrm{Row}\, A)^{\perp}$
$\Rightarrow \mathrm{Nul}\, A^T = (\mathrm{Col}\, A)^{\perp}$

when we take the tranpose, rows become the columns

Dimensions of W and $W^{\perp}$
W = a subspace of $\mathbb{R}^n$
let $\{\vec{v}_1, \dots, \vec{v}_p\}$ be a basis for W
$\Rightarrow W = \mathrm{span}\{\vec{v}_1, \dots, \vec{v}_p\}$

let $A = [\vec{v}_1 \dots \vec{v}_p]$ $\quad A \in \mathbb{R}^{n \times p}$

$W = \mathrm{colsp}\, A$

$\dim W = \dim(\mathrm{colsp}\, A) = \mathrm{rank}\, A = \mathrm{rank}\, A^T$
$\dim W^{\perp} = \dim(\mathrm{colsp}\, A)^{\perp} = \dim \mathrm{Nul}\, A^T$

Recall, rank-nullity theorem
for $B \in \mathbb{R}^{m \times n}$ $\quad F_B : \mathbb{R}^n \to \mathbb{R}^m$
$\mathrm{rank}(B) + \dim(\mathrm{Nul}\, B) = n$

Apply it to A^T $\quad A^T \in \mathbb{R}^{p \times n}$
$\mathrm{rank}(A^T) + \dim(\mathrm{Nul}\, A^T) = n$
$\downarrow$
$\dim(W) + \dim(W^{\perp}) = n$

Ex $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 2 & -1 \end{bmatrix}$

a) Find $\dim\left((\text{Colsp} A)^{\perp}\right)$

b) Find a basis for $(\text{Colsp} A)^{\perp}$

a) Colsp A is ~~a bsp~~ a subspace in $\mathbb{R}^3$

$\dim(\text{Colsp} A) = 2$

$\Rightarrow \dim(\text{Colsp} A)^{\perp} = 3 - \dim(\text{Colsp} A) = 1$

b) $A^T = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & -1 \end{bmatrix} \xrightarrow[\text{row reduction}]{} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

pivot variables x_1, x_3

free variable x_2

Solution $A^T x = 0$

$x_1 + x_2 = 0$

$x_3 = 0$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for $\text{Nul} A^T$

$= (\text{Colsp} A)^{\perp}$

Orthogonal Set

Def: $\{\vec{u}_1, \ldots, \vec{u}_p\}$ is called an orthogonal set if

$$\vec{u}_i \cdot \vec{u}_j = 0 \quad \text{for } i \neq j$$

An orthogonal set of non-zero vectors is linearly independent

Def. If a basis for W is an orthogonal set then it is called an orthogonal basis for W

Ex. $\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $\vec{u}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

Show that $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthogonal basis for $\mathbb{R}^3$

1) check that they are orthogonal

$\vec{u}_1 \cdot \vec{u}_2 = 1 + 1 - 2 = 0$

$\vec{u}_1 \cdot \vec{u}_3 = -1 + 1 + 0 = 0$

$\vec{u}_2 \cdot \vec{u}_3 = -1 + 1 + 0 = 0$

$\Rightarrow \{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthogonal set

$\Rightarrow$ linearly independent

2) Show $\text{Span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3\} = \mathbb{R}^3$

\# of vectors $= 3$

$\dim \mathbb{R}^3 = 3$

and since the 3 vectors are linearly independent they span $\mathbb{R}^3$

$\Rightarrow \{u_1, u_2, u_3\}$ is an orthogonal basis for $\mathbb{R}^3$

> An orthogonal set of n non-zero vectors in $\mathbb{R}^n$ is an orthogonal basis for $\mathbb{R}^n$

<u>Theorem</u> Let $\{\vec{u}_1, \dots, \vec{u}_p\}$ be an orthogonal basis for W

Each $\vec{y}$ in W can be represented as

$$\vec{y} = c_1 \vec{u}_1 + \dots + c_p \vec{u}_p$$

The coefficients $\{c_j\}$ are given by

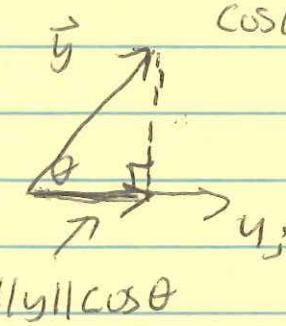
$$\boxed{c_j = \frac{\vec{y}\cdot\vec{u}_j}{\underbrace{\vec{u}_j\cdot\vec{u}_j}_{\|u\|^2}}}$$

projection

normalization factor

Recall in 2D

$\vec{y}\cdot\vec{u}_j = \|y\|\|u_j\| \cos(\theta)$



Ex

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \vec{u}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{y} = \begin{bmatrix} 6 \\ 2 \\ 5 \end{bmatrix}$$

$$\vec{y} = c_1 \vec{u}_1 + c_2 \vec{u}_2 + c_3 \vec{u}_3$$

$$c_1 = \frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} = \frac{6 + 2 + 10}{1 + 1 + 4} = \frac{18}{6} = 3$$

$$c_2 = \frac{\vec{y} \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2} = \frac{6 + 2 - 5}{1 + 1 + 1} = \frac{3}{3} = 1$$

$$c_3 = \frac{\vec{y} \cdot \vec{u}_3}{\vec{u}_3 \cdot \vec{u}_3} = \frac{-6 + 2 + 0}{1 + 1 + 0} = \frac{-4}{2} = -2$$

$$(\vec{y} = c_1 \vec{u}_1 + c_2 \vec{u}_2 + c_3 \vec{u}_3) \cdot \vec{u}_2$$

$$\underline{\underline{\vec{y} \cdot \vec{u}_2}} = c_1 \underbrace{\vec{u}_1 \cdot \vec{u}_2}_{=0} + c_2 \underline{\underline{\vec{u}_2 \cdot \vec{u}_2}} + c_3 \underbrace{\vec{u}_3 \cdot \vec{u}_2}_{=0}$$

$$c_2 = \frac{\vec{y} \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2}$$

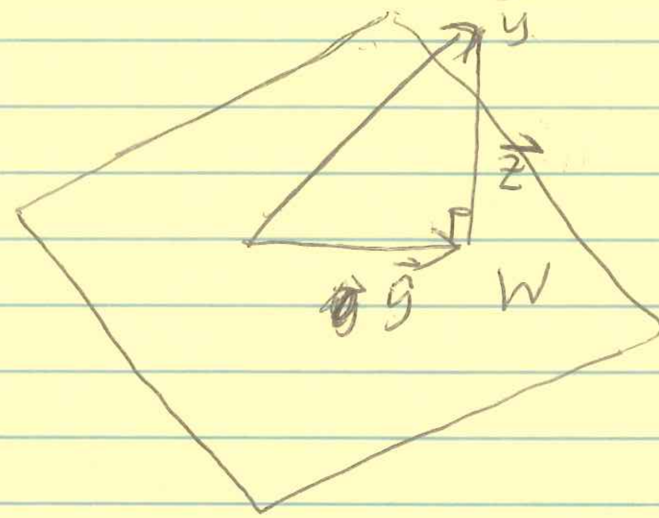
Orthogonal projection

Theorem

(orthogonal decomposition theorem)

Let W be a subspace of $\mathbb{R}^n$

Let $\{\vec{u}_1, \dots, \vec{u}_p\}$ be an orthogonal basis for W



Each $\vec{y}$ in $\mathbb{R}^n$ can be decomposed as $\vec{y} = \hat{y} + \vec{z}$

where $\hat{y}$ is in W and $\vec{z}$ is in $W^{\perp}$

One solution for $\hat{y}$ is given by

$$\hat{y} = \left(\frac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\right)\vec{u}_1 + \dots + \left(\frac{\vec{y}\cdot\vec{u}_p}{\vec{u}_p\cdot\vec{u}_p}\right)\vec{u}_p$$

$\hat{y}$ is unique

$\hat{y}$ is called the orthogonal projection of $\vec{y}$ onto W

Notation $\text{proj}_W \vec{y}$

Theorem (Best representation theorem)
Let W be a subspace of $\mathbb{R}^n$
Let $\vec{y}$ be a vector in $\mathbb{R}^n$

We have

$$\| \vec{y} - \text{proj}_W \vec{y} \| < \| \vec{y} - \vec{v} \|$$

for all $\vec{v} \in W$ and
$\vec{v} \neq \text{proj}_W \vec{y}$ in W