Lecture 17

Diagonalization of symmetric matrices and The Gram-Schmidt Process

Def: If $A^T = A$, we say A is symmetric

Ex 1:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \text{ is symmetric}$$

$A^T = A$ if and only if $a_{ij} = a_{ji}$

$[A]_{ij} = a_{ij}$

Ex 2:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 3 \end{bmatrix} \text{ is not symmetric}$$

Ex I:

$$A^T = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = A$$

Statements: If A is symmetric then all eigenvalues are real.

Theorem: If A is symmetric, then eigenvectors for different eigenvalue are orthogonal to each other for orthogonality $\vec{v}_1 \cdot \vec{v}_2 = 0$

proof: $A\vec{v}_1 = \tau_1 \vec{v}_1$

$A\vec{v}_2 = \tau_2 \vec{v}_2$, $\tau_1 \neq \tau_2$

$\vec{v}_1 \cdot \vec{v}_2$

$\vec{v}_1^T \vec{v}_2$

$$\underbrace{(A\vec{v}_1) \cdot \vec{v}_2}_{\tau_1 \vec{v}_1 \cdot \vec{v}_2} = (A\vec{v}_1)^T \vec{v}_2 = (\vec{v}_1^T A^T)\vec{v}_2$$

$\vec{v}_1 \cdot \vec{v}_2 = \vec{v}_1^T \vec{v}_2$

$(AB)^T = B^T A^T$

$$= \vec{v}_1^T (A^T \vec{v}_2)$$

$$= \vec{v}_1 \cdot (A^T \vec{v}_2)$$

$A^T = A$

$$= \vec{v}_1 \cdot (A\vec{v}_2)$$

$$= \vec{v}_1 \cdot (\tau_2 \vec{v}_2)$$

$$= \tau_2 \vec{v}_1 \cdot \vec{v}_2$$

$$\Rightarrow \tau_1 \vec{v}_1 \cdot \vec{v}_2 = \tau_2 \vec{v}_1 \cdot \vec{v}_2$$

$$\Rightarrow \tau_1 \underbrace{\vec{v}_1 \cdot \vec{v}_2} - \tau_2 \underbrace{\vec{v}_1 \cdot \vec{v}_2} = 0$$

$$(\tau_1 - \tau_2) \underbrace{\vec{v}_1 \cdot \vec{v}_2} = 0$$

$\tau_1 \neq \tau_2 \quad \Rightarrow \quad \vec{v}_1 \cdot \vec{v}_2 = 0 \quad \checkmark$

Suppose an $n \times n$ symmetric matrix A has n distinct eigenvalues

$\tau_1, \tau_2, \ldots, \tau_n$

Eigenvector: $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$

$\Rightarrow \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is an orthogonal set

$\Rightarrow$ linearly independent

$\Rightarrow$ A is diagonalizable by

$$P = [\vec{v}_1, \vec{v}_2, \vec{v}_3, \dots, \vec{v}_n]$$

Def: If $P^{-1} = P^T$, we say P is an orthogonal matrix

We can construct and orthogonal matrix $\tilde{P}$

$\Rightarrow$ Make $\{\vec{v}_1, \dots, \vec{v}_n\}$ an orthonormal set

$$\vec{u}_i = \frac{1}{\|\vec{v}_i\|}\vec{v}_i$$

$$\tilde{P} = [\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n]$$

$$\tilde{P}^T\tilde{P} = \begin{bmatrix} \vec{u}_1^T \\ \vec{u}_2^T \\ \vdots \\ \vec{u}_n^T \end{bmatrix} [\vec{u}_1\ \vec{u}_2 \dots \vec{u}_n] \quad \in \mathbb{R}^{n\times n}$$

$$= \begin{bmatrix} \vec{u}_1^T\vec{u}_1 & \vec{u}_1^T\vec{u}_2 & \dots & \vec{u}_1^T\vec{u}_n \\ \vec{u}_2^T\vec{u}_1 & \vec{u}_2^T\vec{u}_2 & \dots & \\ \vdots & & \ddots & \\ \vec{u}_n^T\vec{u}_1 & & & \vec{u}_n^T\vec{u}_n \end{bmatrix} \quad =0 \quad =1$$

$$= \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & 1 & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix}$$

Def : We say A is <u>orthogonally diagonalizable</u> if there exists an orthogonal matrix P s.t. $P^T A P = D$

$\Rightarrow$ If an $n \times n$ symmetric matrix A has n distinct eigenvalues, Then A is <u>orthogonally</u> diagonalizable

Theorem

<u>Every</u> symmetric matrix is orthogonally diagonalizable

<u>The Spectral Theorem for Symmetric Matrices</u>

An $n \times n$ symmetric matrix A has the following properties

a) A has n real eigenvalues, counting mutiplicities

b) ~~Fore~~ For repeated eigenvalues geometric multiplicity = algebraic multiplicity

c) The eigenspaces are mutually orthogonal

d) A is orthogonally diagonalizable

Consider $A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$

$p(s) = \det(sI - A)$
3rd order polynomial
$\Rightarrow$ 3 roots

$\tau_{1,2} = 7$ (algebraic multiplicity $= 2$), $\tau_3 = -2$

$E_{\tau=7} = \text{Span}\left\{ \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}_{\vec{v}_1}, \underbrace{\begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}}_{\vec{v}_2} \right\}$ $\quad E_{\tau=-2} = \text{Span}\left\{ \begin{bmatrix} -1 \\ -1/2 \\ 1 \end{bmatrix} \right\}$

$(A - \tau I)X = 0$

$\uparrow$ eigenvectors

$\dim(E_{\tau=7}) = 2$ (geometric multiplicity $= 2$)

Any eigenvector for $\tau = 7$ can be expressed as linear combination

$V = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2$

$= \beta_1 \vec{u}_1 + \beta_2 \vec{u}_2$

$\text{Span}\{\vec{u}_1, \vec{u}_2\} = \text{Span}\{\vec{v}_1, \vec{v}_2\}$

$\Rightarrow$ Make $\vec{u}_1, \vec{u}_2$ orthonormal
the we can construct matrix P

$P = [\vec{u}_1, \vec{u}_2, \vec{v}_3]$

The Gram-Schmidt Process

Given a basis $\{\vec{x}_1, \dots, \vec{x}_p\}$ for a nonzero subspace W of $\mathbb{R}^n$ define

1^{st} vector

$$\vec{v}_1 = \vec{x}_1 \qquad U = \text{span}\{x_1\} = \text{span}\{v_1\}$$

$$\vec{v}_2 = \vec{x}_2 - \underbrace{\frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1}_{\text{projection of } \vec{x}_2 \text{ onto } U}$$

$$\vec{v}_3 = \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 - \frac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2$$

$$\vdots$$

$$\vec{v}_p = \vec{x}_p - \frac{\vec{x}_p \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 - \frac{\vec{x}_p \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2 - \dots - \frac{\vec{x}_p \cdot \vec{v}_{p-1}}{\vec{v}_{p-1} \cdot \vec{v}_{p-1}}\vec{v}_{p-1}$$

$$\text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\} = \text{span}\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_p\}$$

$\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\}$ is an orthogonal basis for W

$$\Rightarrow \text{span}\{\vec{v}_1, \dots, \vec{v}_k\} = \text{span}\{\vec{x}_1, \dots, \vec{x}_k\}$$

for $1 \le k \le p$

Let

$$\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$$W = \text{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$$

Construct an orthogonal basis for W

set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

$$\vec{v}_1 = \vec{x}_1 \qquad W_1 = \text{span}\{v_1\} = \text{span}\{x_1\}$$

$$\vec{v}_2 = \vec{x}_2 - \text{proj}_{W_1} x_2$$

$$= \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_2} v_1 = \begin{bmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} \text{ equiv } \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$W_2 = \text{span}\{v_1, v_2\} = \text{span}\{x_1, x_2\}$$

$$\vec{v}_3 = \vec{x}_3 - \text{proj}_{W_2} \vec{x}_3$$

$$= \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 - \frac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2$$

$$= \begin{bmatrix} 0 \\ -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

To find an orthonormal basis just normalize

$$\tilde{v}_1 = \frac{1}{\|\vec{v}_1\|} v_1, \quad \tilde{v}_2 = \frac{1}{\|\vec{v}_2\|} \vec{v}_2, \quad \tilde{v}_3 = \frac{1}{\|\vec{v}_3\|} \vec{v}_3$$

Revisit diagonalization problem on pg. 5

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{z}_1 = \vec{v}_1$$

$$\vec{z}_2 = \vec{v}_2 - \frac{\vec{v}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} v_1 = \begin{bmatrix} -1/4 \\ 1 \\ 1/4 \end{bmatrix}$$

$\Rightarrow$ 3 orthogonal vectors

$$E_{\tau=7} = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1/4 \\ 1 \\ 1/4 \end{bmatrix} \right\}$$

$$E_{\tau=2} = \text{span}\left\{ \begin{bmatrix} -1 \\ -1/2 \\ 1 \end{bmatrix} \right\}$$