Algebraic Multiplicity vs. Geometric Multiplicity

Ex 1. $A = \begin{bmatrix} 3 & 2 & -4 & 1 \\ 0 & 1 & 8 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\lambda_1 = 3 \quad \lambda_2 = 1 \quad \lambda_3 = -2 \quad \lambda_4 = 1$

repeated eigenvalue

$\lambda_{2,4} = 1$: algebraic multiplicity of two

λ_1, λ_3 : algebraic multiplicity of one

$\Rightarrow$ guaranteed geometric multiplicity of one

$\lambda = 1 \Rightarrow E_{\lambda=1} = \text{Span}\left\{ \begin{bmatrix} -.7071 \\ 0.7071 \\ 0 \end{bmatrix} \right\}$

$\dim(E_{\lambda=1}) = 1 \leftarrow$ geometric multiplicity: The dimension of the associated eigenspace

Ex 2

$$A = \begin{bmatrix} 4 & 2 & 3 & 3 \\ 0 & 2 & \alpha & 3 \\ 0 & 0 & 4 & 14 \\ 0 & 0 & 0 & 2 \end{bmatrix} \qquad \begin{aligned} \tau_1 &= 4 \\ \tau_2 &= 2 \\ \tau_3 &= 4 \\ \tau_4 &= 2 \end{aligned}$$

Find the values of α in matrix A s.t. the geometric multiplicity of $\tau = 4$ is 2.

- We want to look at solutions for $(A-4I)x = 0$

$$A - 4I = \begin{bmatrix} 0 & 2 & 3 & 3 \\ 0 & -2 & \alpha & 3 \\ 0 & 0 & 0 & 14 \\ 0 & 0 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 0 & 2 & 3 & 3 \\ 0 & 0 & \alpha+3 & 6 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

If $\alpha = -3$, we will have two free variables x_1 and x_3

$$\Rightarrow \quad \begin{aligned} 2x_2 + 3x_3 + 3x_4 &= 0 \\ 6x_4 &= 0 \\ x_4 &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_4 &= 0 \\ x_2 &= -\tfrac{3}{2}x_3 \end{aligned}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ -3/2 \\ 1 \\ 0 \end{bmatrix} x_3$$

$$E_{\tau=4} = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -3/2 \\ 1 \\ 0 \end{bmatrix} \right\} \Rightarrow \dim(E_{\tau=4}) = 2$$

geometric multiplicity