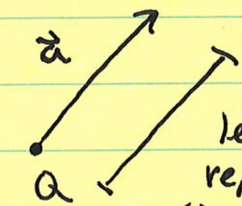
Lecture 1 Vectors and complex numbers

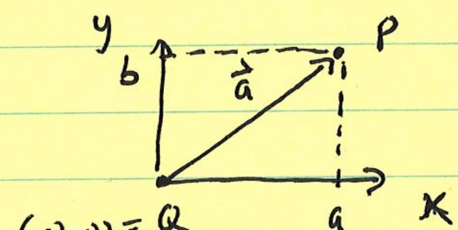
Vectors are represented as arrows

Vector acts at point Q



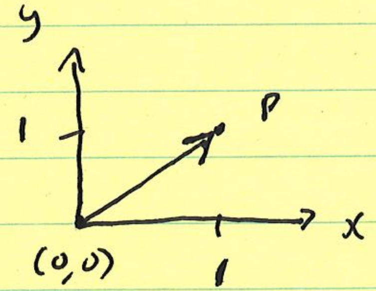
length represents the magnitude

It is common to use a reference coordinate system to define vectors. Consider the following system in 2-D space w.r.t. p.t. Q



Assume $\vec{a}$ has endpoint $P = (a, b)$

Let $P = (1,1)$



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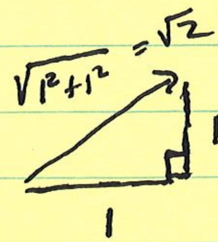
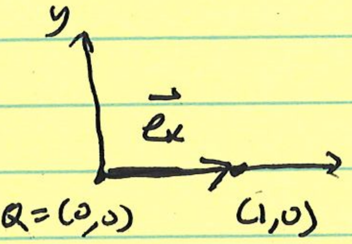


length vector or magnitude denoted by $|\vec{a}|$

Let Q be the origin (0,0), then the vector is completely determined by its end point

We can define a unit vector by its end point $(1,0)$ along the x-axis as

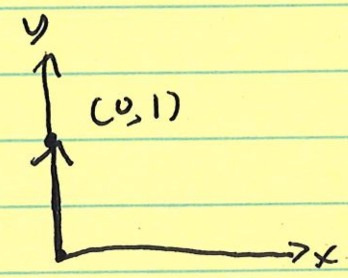
$$\vec{e}_x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$



$$|\vec{e}_x| = 1$$

Simarly, we define a unit vector along the y-axis as

$$\vec{e}_y = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



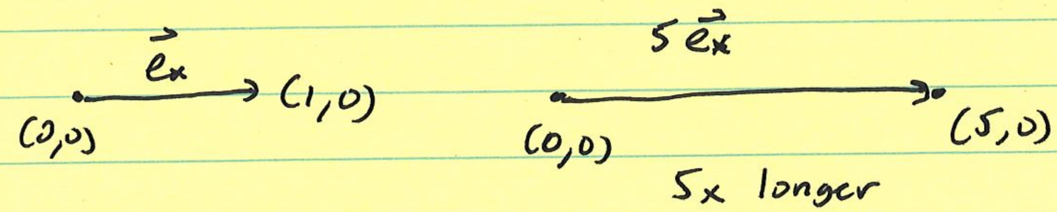
$$|\vec{e}_y| = 1$$

with end point $(0,1)$

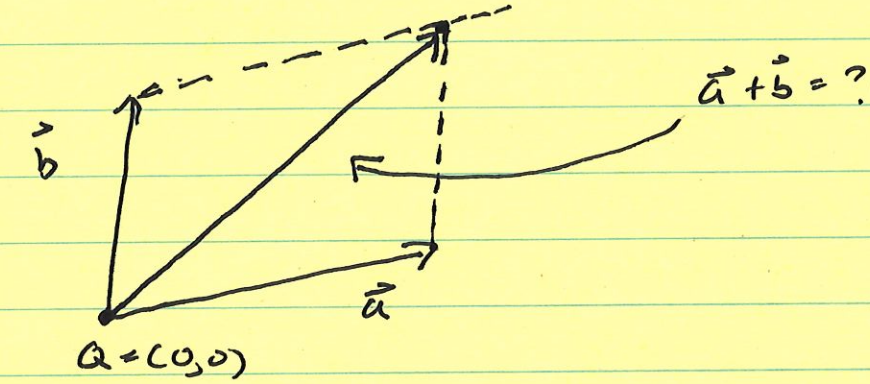
Then $\vec{a}$ can be represented as a linear combination of the two, that is

$$\vec{a} = \alpha_x \vec{e}_x + \alpha_y \vec{e}_y \qquad \alpha_x, \alpha_y \in \mathbb{R}$$

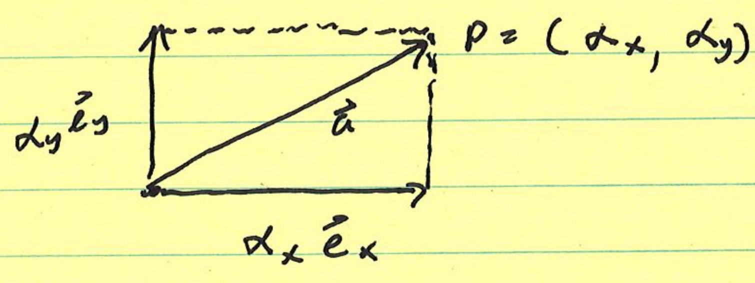
Ex: $$\alpha_x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \alpha_y \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \vec{a}$$



Parallelogram law for vector addition: The sum of two vectors $\vec{a}$ and $\vec{b}$ that act at the same point Q is the vector In the parallelogram having $\vec{a}$ and $\vec{b}$ as adjacent sides that is represented by a diagonal beginning at Q



So what is $\vec{a} = \alpha_x \vec{e}_x + \alpha_y \vec{e}_y$?

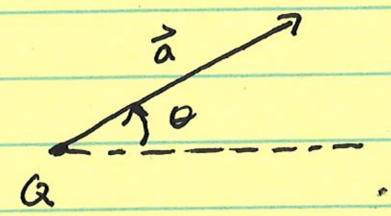


By the parallelogram law of addition of vectors, it follows that summing two vectors $\vec{a} + \vec{b}$ is a component wise operation.

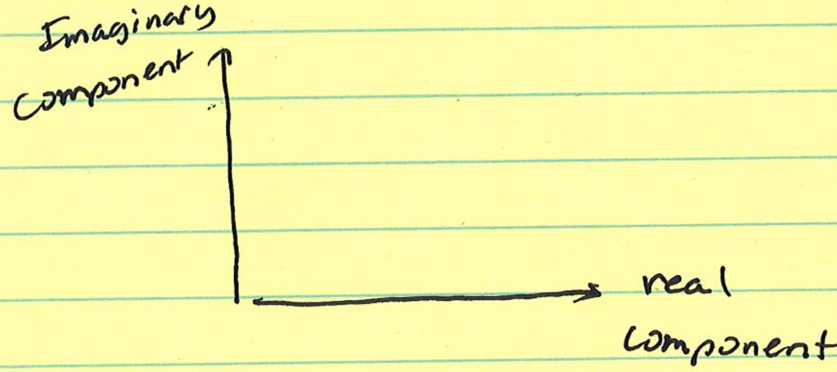
Ex $\vec{a} + \vec{b} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \end{bmatrix}$

Note we can use polar coordinates to define vector (r, θ)

$r = |\vec{a}|$



Consider a complex reference coordinate system



The imaginary component is the scalar multiplying the quantity called "i" where $i = \sqrt{-1}$ s.t. $i^2 = -1$

Ex: Quadratic formula

$$ax^2 + bx + c = 0$$

Solution $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

case: $b^2 < 4ac$ $\Rightarrow$ gives imaginary component

$$x = \frac{-b \pm \sqrt{\overbrace{(-1)}^{i}(-b^2+4ac)}}{2a} = \frac{-b \pm i\sqrt{-b^2+4ac}}{2a}$$

$$= \underbrace{\overbrace{\frac{-b}{2a}}^{\text{real \#}}}_{\text{real component}} \pm \underbrace{\overbrace{\frac{\sqrt{-b^2+4ac}}{2a}}^{\text{real \#}}\, i}_{\text{Imaginary component}}$$

This is a complex number $\mathbb{C}$

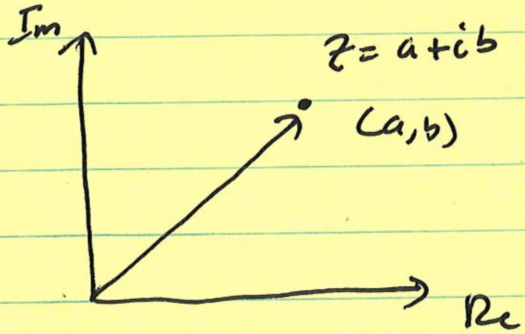
Def: A complex number is an expression of the form $a+bi$ where a and b are real numbers ($\mathbb{R}$)

$a+bi = c+di$ iff $a=c$ and $b=d$

"if and only if"

We can use a complex coordinate system to represent complex numbers, this is called a complex plane

$z \in \mathbb{C}$



Multiplication of complex numbers

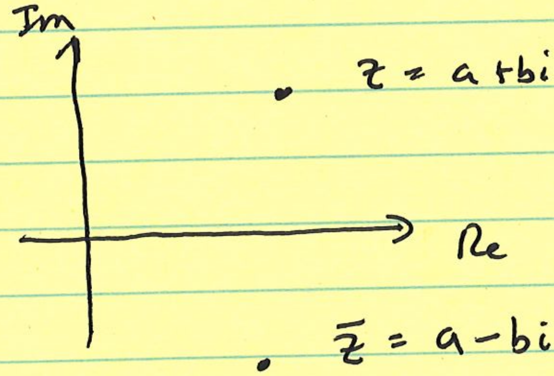
$(a+ib)(c+di)$ Apply FOIL
First
Outer
Inner
Last

$$= ac + adi + ibc + \underline{ib(di)}$$
$$= ac + (ad+bc)i + bd\underbrace{i^2}_{-1}$$
$$\underline{\hphantom{bd\,i^2}}$$
real #

$= \cancel{ac}$

$$= \underbrace{ac - bd}_{\text{real component}} + \underbrace{(ad+bc)i}_{\text{imaginary component}}$$

Returning to complex plane



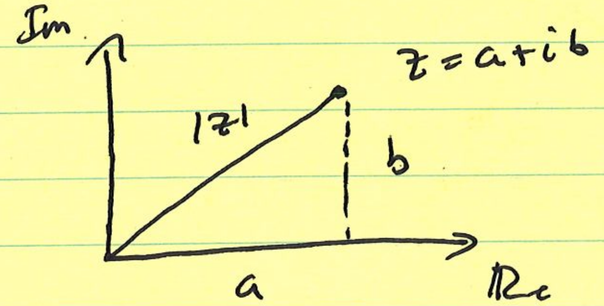
Reflection across the real axis is called the complex conjugate

Note that if $z = \bar{z} \Rightarrow z \in \mathbb{R}$

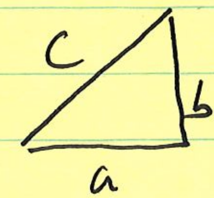
Another property of complex numbers is the absolute value or modulus

Def: The absolute value of a number $z = a + bi$ is defined by $|z|$ and is given by

$$|z| = \sqrt{a^2 + b^2}$$



$$a^2 + b^2 = c^2$$



$$c = \sqrt{a^2 + b^2}$$