Lecture 1  Vectors and Complex numbers

Vectors are represented as arrows

\[ \vec{a} \]

at point Q

\[ \text{length represents the magnitude} \]

It is common to use a reference coordinate system to define vectors. Consider the following system in 2-D space w.r.t. point Q

\[ (0,0) = Q \]

\[ \hat{a} \]

Assume \( \hat{a} \) has endpoint \( P = (a, b) \)

Let \( P = (1, 1) \)

\[ \sqrt{1^2 + 1^2} = \sqrt{2} \]

length vector or magnitude denoted by \( |\hat{a}| \)

Let \( Q \) be the origin \((0,0)\), then the vector is completely determined by its endpoint.
We can define a unit vector by its end point \((1,0)\) along the \(x\)-axis as

\[
\vec{e}_x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\]

\[|\vec{e}_x| = 1\]

Similarly, we define a unit vector along the \(y\)-axis as

\[
\vec{e}_y = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

\[|\vec{e}_y| = 1\]

with end point \((0,1)\).

Then \(\vec{a}\) can be represented as a linear combination of the two, that is

\[
\vec{a} = a_x \vec{e}_x + a_y \vec{e}_y, \quad a_x, a_y \in \mathbb{R}
\]

Ex: \[a_x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + a_y \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \vec{a}\]

\[
\begin{align*}
\vec{e}_x & \quad \text{from } (0,0) \quad \text{to } (1,0) \\
5 \vec{e}_x & \quad \text{from } (0,0) \quad \text{to } (5,0) \quad 5x \text{ longer}
\end{align*}
\]
Parallelogram law for vector addition: The sum of two vectors $\vec{a}$ and $\vec{b}$ that act at the same point $Q$ is the vector in the parallelogram having $\vec{a}$ and $\vec{b}$ as adjacent sides that is represented by a diagonal beginning at $Q$.

So what is $\vec{a} = ax \hat{e}_x + ay \hat{e}_y$?

By the parallelogram law of addition of vectors, it follows that summing two vectors $\vec{a} + \vec{b}$ is a component wise operation.

Ex $\vec{a} + \vec{b} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \end{bmatrix}$
Note we can use polar coordinates to define vector \((r, \theta)\).

\[ \hat{a} \]

\[ r \to 1 \hat{a} \]

Consider a complex reference coordinate system.

The imaginary component is the scalar multiplying the quantity called "i" where \(i = \sqrt{-1}\) s.t. \(i^2 = -1\).

Ex: Quadratic formula
\[ ax^2 + bx + c = 0 \]

Solution
\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

Case: \(b^2 < 4ac\) \(\Rightarrow\) gives imaginary component.
\[ x = \frac{-b \pm \sqrt{(-1)(-b^2+4ac)}}{2a} = -b \pm \frac{i \sqrt{-b^2+4ac}}{2a} \]

This is a complex number \( i \) 

**Def:** A complex number is an expression of the form \( a+bi \) where \( a \) and \( b \) are real numbers \((\mathbb{R})\)

\[ a+bi = c+di \iff a=c \text{ and } b=d \]

"if and only if"

We can use a complex coordinate system to represent complex numbers, this is called a complex plane

\[ z = a+ib \quad z \in \mathbb{C} \]
Multiplication of complex numbers

\[(a + bi)(c + di)\]

Apply FOIL

First

Outer

Inner

Last

\[= ac + ad\overline{i} + bci + bd\overline{(di)}\]

\[= ac + (ad + bc)i + bd\overline{i^2}\]

\[= ac + (ad + bc)i + bd(-1)\]

\[= ac - bd + (ad + bc)i\]

real component

imaginary component

Returning to complex plane

\[\text{Im} \quad -\quad z = a + bi\]

\[\text{Re} \quad \overline{z} = a - bi\]

Reflection across the real axis is called the complex conjugate

Note that if \(z = \overline{z} \Rightarrow z \in \mathbb{R}\)
Another property of complex numbers is the absolute value or modulus.

Def: The absolute value of a number \( z = a + bi \) is defined by

\[ |z| = \sqrt{a^2 + b^2} \]

\[ a^2 + b^2 = c^2 \]

\[ c = \sqrt{a^2 + b^2} \]