Lecture 2 Complex numbers continued

Last lecture we introduced the imaginary number i and showed that this number can emerge when solving

$$ax^2 + bx + c = 0$$

This has the solution

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

When $b^2 < 4ac$ we get a negative inside the square root

$$x = \frac{-b \pm \sqrt{(-1)(-b^2+4ac)}}{2a} = \frac{-b \pm \overbrace{\sqrt{-1}}^{i} \cdot \sqrt{-b^2+4ac}}{2a}$$

$$= \underbrace{\overbrace{-\frac{b}{2a}}^{\text{real \#}}}_{\substack{\text{real} \\ \text{Component}}} \pm \underbrace{\overbrace{\frac{\sqrt{-b^2+4ac}}{2a}}^{\text{real \#}}}_{\substack{\text{Imaginary} \\ \text{component}}} i$$

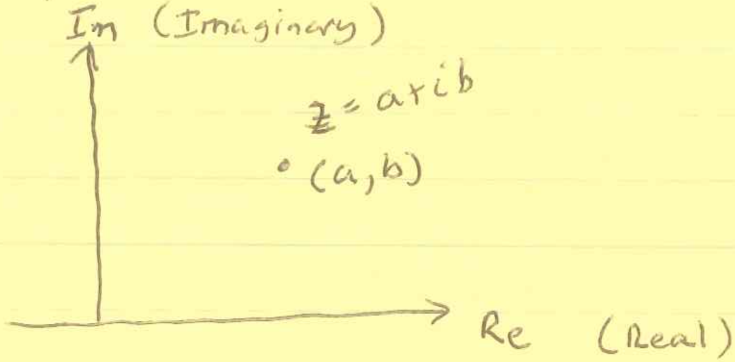
This is a complex number

Def: A complex number is an expression of the form $a+bi$ where a and b are real numbers

$a+bi = c+di$ iff $a=c$ and $b=d$

"if and only if"

We can use a complex coordinate system to represent complex numbers, this is called a complex plane



We refer to the point that represents the complex number z as "the point z"
the point $z = a + bi$ is the point with coordinates (a,b)

Recall, the $x-y$ coordinate system allowed us to decompose a vector into components in the x and y direction

In this case we are decomposing a number into real and imag parts

Like adding Apples and Oranges

5 apples + 3 oranges	+	2 apples + 6 oranges	=	7 apples + 9 oranges

How about multiplication?

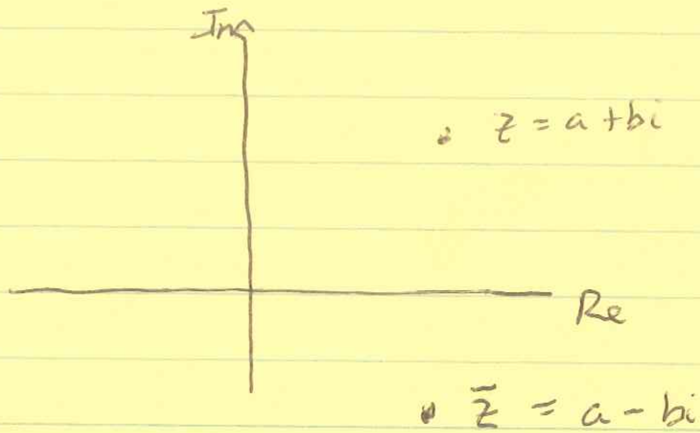
$(a+bi)(c+di)$ Apply FOIL

First
Outer
Inner
Last

$ac + adi + cbi + bd\underbrace{i^2}_{-1}$

$= \underbrace{(ac-bd)}_{Re} + \underbrace{(ad+cb)}_{Im} i$

Returning to complex plane



The reflection across the real axis is called the complex conjugate

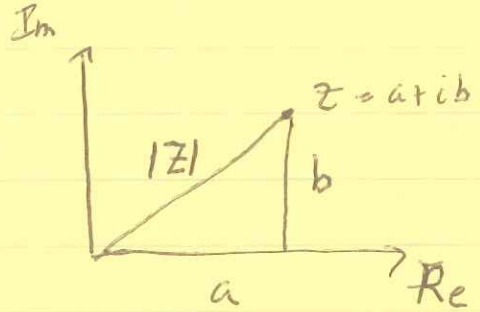
Note that if $z = \bar{z}$ if $z \in \mathbb{R}$ $\mathbb{R}$: real numbers

Complex conjugates play a significant role in the theory of complex variables

Another property of complex numbers is the absolute value or modulus

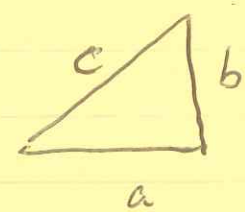
Def: The absolute value or modulus of a number $z = a + bi$ is denoted by $|z|$ and is given by

$$|z| = \sqrt{a^2 + b^2}$$



We use the pythagorean theorem

$$a^2 + b^2 = c^2$$

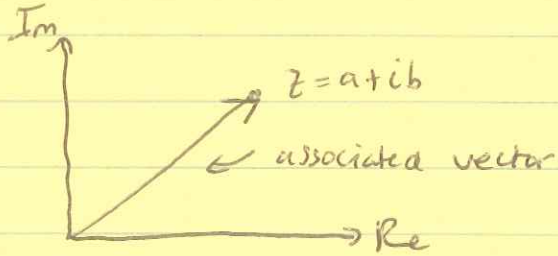


Note $|z|$ is always a nonnegative real number

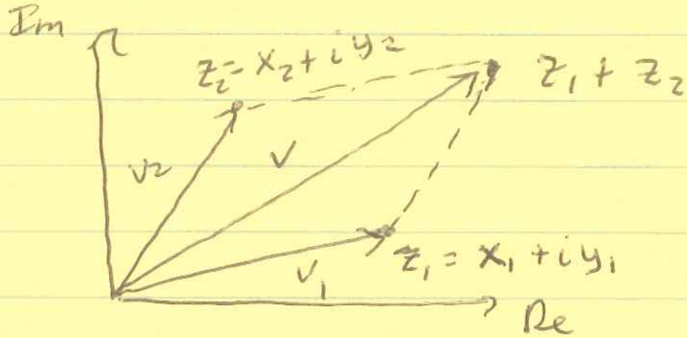
What happens if we multiply z by $\bar{z}$?

$$z\bar{z} = (a+ib)(a-ib) = a^2 + \cancel{iab} - \cancel{iab} + b^2$$
$$= a^2 + b^2 = |z|^2$$

With each point z in the complex plane we can associate a vector



Suppose we want to add $V = V_1 + V_2$, we can apply the parallelogram law



If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

then $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$

V has the end point $(x_1 + x_2, y_1 + y_2)$

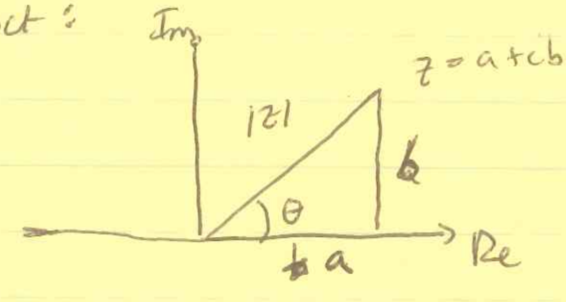
Thus, the correspondance between complex numbers and planar vectors carries over the operation of addition.

We have that $|z|$ is the length of the vector

Then all we need is a direction to define the complex number

We found how to calculate the length, how do we define the direction ?

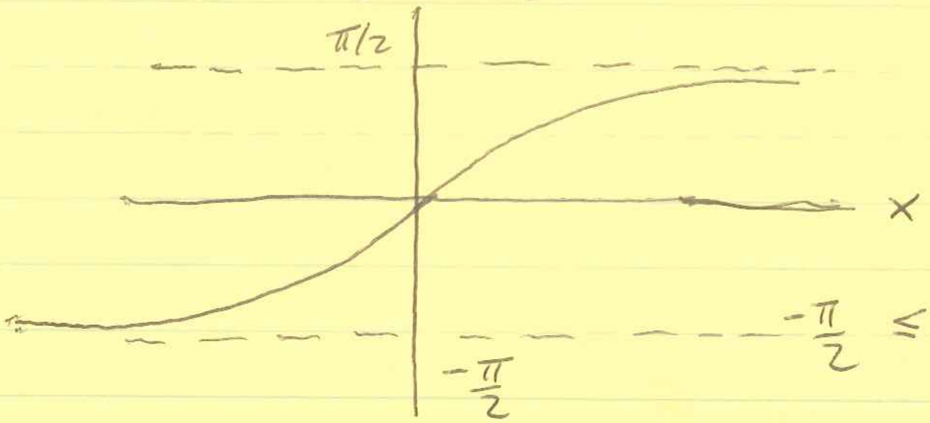
First instinct :



$\tan\theta = \dfrac{b}{a}$

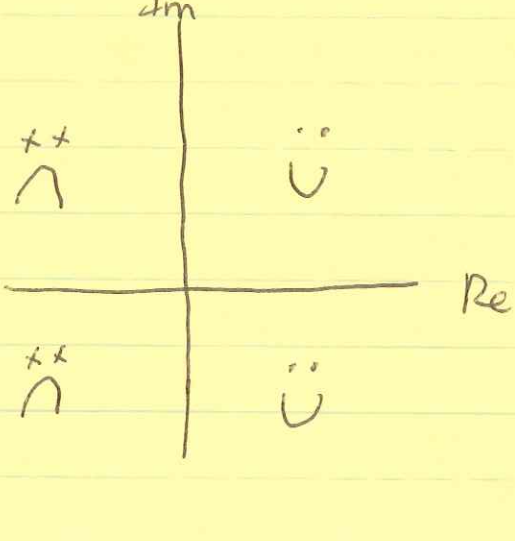
$\theta = \cancel{\arctan\left(\frac{a}{b}\right)}$ $\theta = \arctan\left(\dfrac{b}{a}\right)$

$\theta = \arctan(x)$



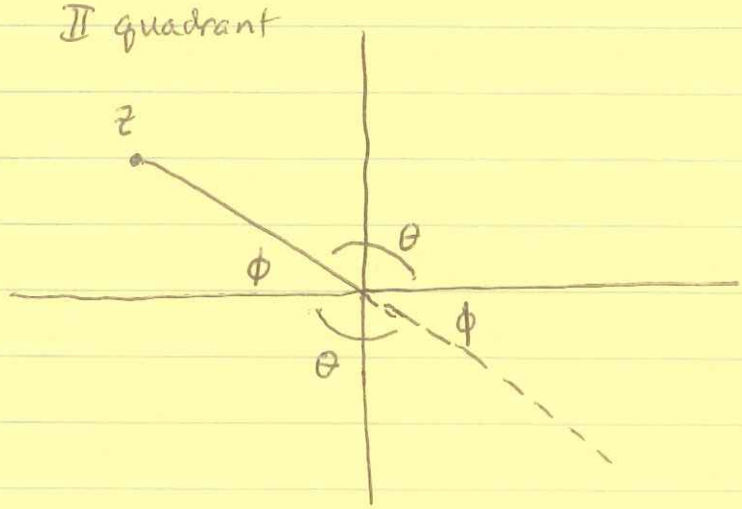
$\frac{\pi}{2} = 90^\circ$

$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$



How do we fix this problem? $z = a + ib$

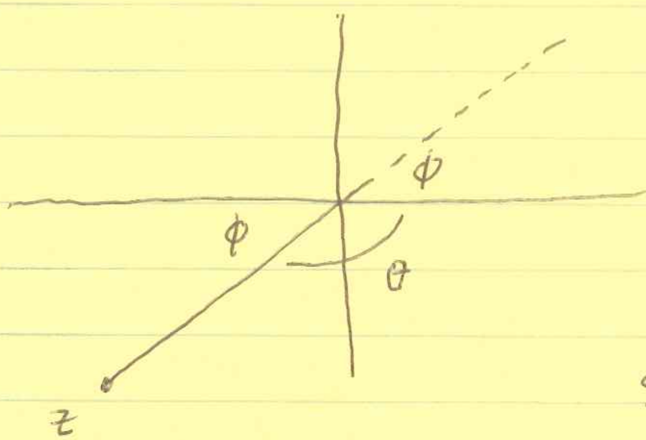
II quadrant



$$\phi = \arctan\left(\frac{b}{a}\right)$$

$$\theta = \phi + \pi$$

Add π to ϕ to get the right angle θ



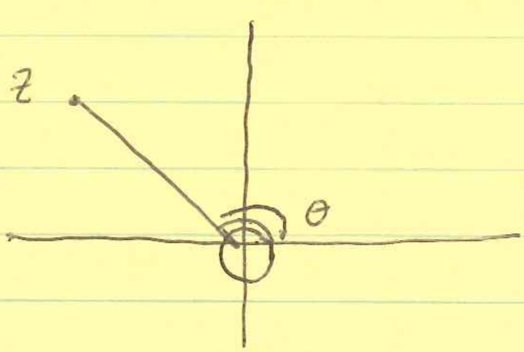
$$\phi = \arctan\left(\frac{b}{a}\right)$$

$$\theta = \phi - \pi$$

Subtract π from ϕ to get the right angle θ

III quadrant

One more issue:



If θ is a defining angle the so is

$\theta + 2\pi$

and $\theta + 4\pi$

and $\theta + 6\pi$

$\vdots$

We call the value of the angle of z $\arg z$ or phase of z. If θ_0 is a value of $\arg z$, then so is

$$\theta_0 \pm 2\pi k \quad \text{for } k = 0, \pm 1, \pm 2, \dots$$

We now introduce the complex exponential

$$\frac{d}{dx} e^{ax} = e^{ax} a$$

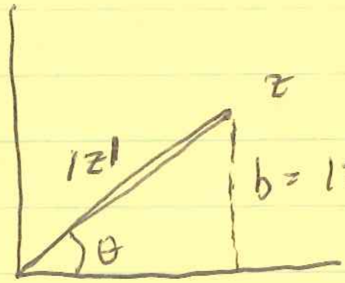
$$\boxed{e^{i\theta} = \cos\theta + i \sin\theta}$$

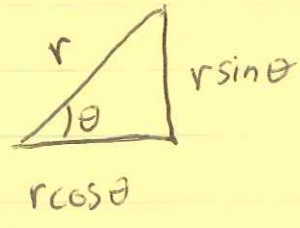
$$\frac{d}{d\theta} e^{i\theta} = i e^{i\theta} = i \cos\theta - \sin\theta$$

$$\frac{d^2}{d\theta^2} e^{i\theta} = i^2 e^{i\theta} = -e^{i\theta} = -\cos\theta - i \sin\theta$$

Complex exponentials make life easier!!!

Consider again the complex number z on the complex plane





$a = |z| \cos\theta$

$$z = a + ib = |z| \cos\theta + i |z| \sin\theta = |z| (\cos\theta + i \sin\theta)$$
$$= |z| e^{i\theta}$$

Example

Calculate z^3

$$(a+ib)^3 \quad \text{vs} \quad (|z|e^{i\theta})^3 = |z|^3 e^{i\theta 3}$$

Calculate $z^{1/3}$

$$(a+ib)^{1/3} \quad \text{vs} \quad (|z|e^{i\theta})^{1/3} = |z|^{1/3} e^{i\theta/3}$$