Lecture 2

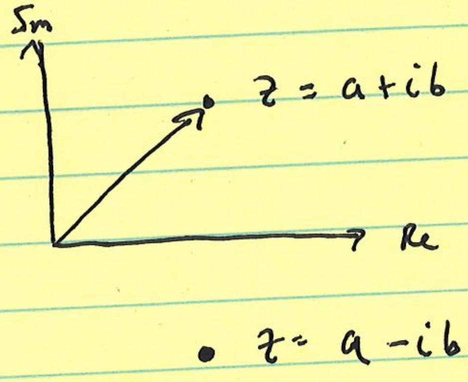
- concept of vectors
- vector addition
- complex numbers

$|z|, \ \bar{z}, \ z_1 \cdot z_2, \ i$

$\begin{bmatrix}1\\2\end{bmatrix} \in \mathbb{R}^2$

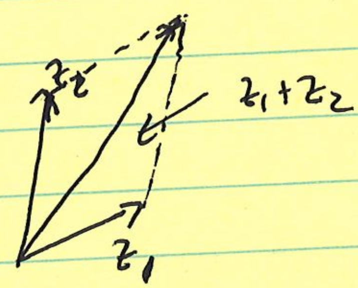
TH Q1

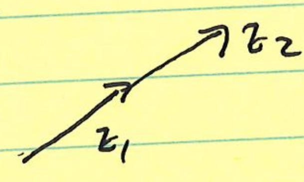
$a, b \in \mathbb{R}$



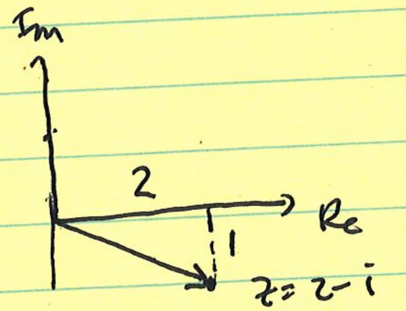
$|z_1 + z_2| \leq |z_1| + |z_2|$

TH Q2





TH Q3



$|z| = \sqrt{2^2 + (-1)^2}$

$= \sqrt{4+1} = \sqrt{5}$

• $i = \sqrt{-1}, \ i^2 = -1$

TH Q4 $\quad z = i$

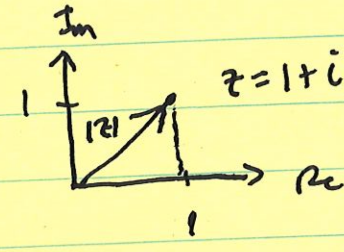
$z\bar{z} = (i)(-i) = -i^2 = -(-1) = 1$

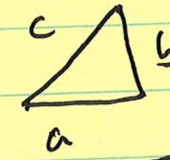
$|z|^2 = z\bar{z}$

TH Q5 'complex conjugate

$z = 1+i \qquad \bar{z} = 1-i$

$z\bar{z} = (1+i)(1-i) = 1 - \cancel{i} + \cancel{i} + 1 = 2$





$c = \sqrt{a^2+b^2}$

$a=1,\ b=1 \qquad |z| = \sqrt{1^2+1^2}$

$= \sqrt{2}$

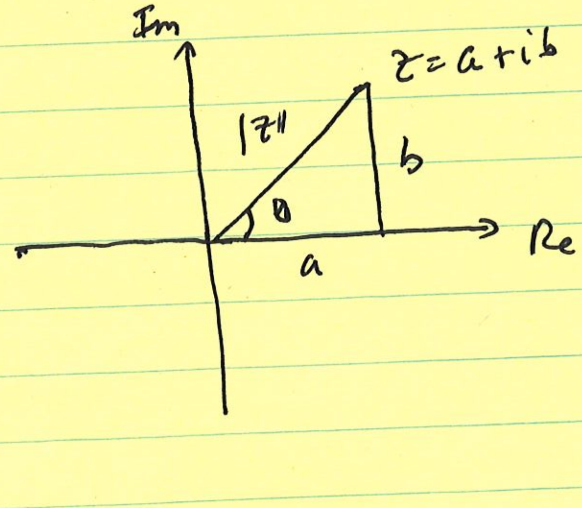
$z\bar{z} = |z|^2$

$z\bar{z} = (a+ib)(a-ib) = a^2 + \cancel{iab} - \cancel{iab} + b^2$

$= a^2 + b^2 = |z|^2$

How do we define the direction of a vector associated with complex number

$z = a+ib$

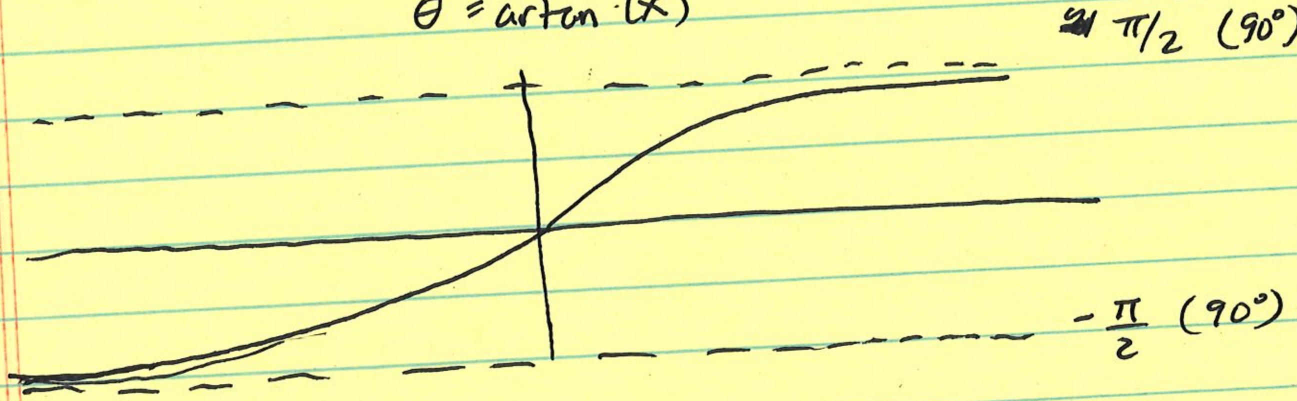


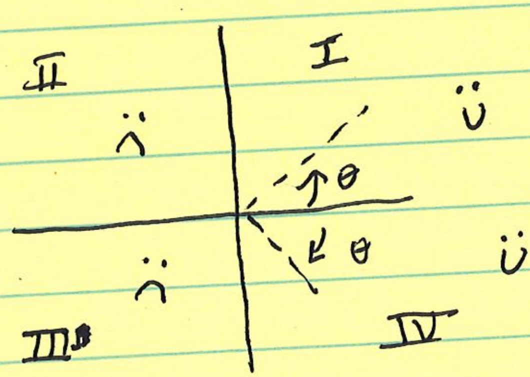
$\tan\theta = \frac{b}{a}$

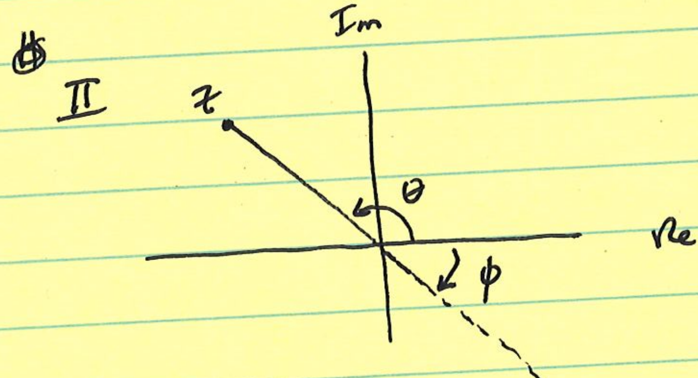
$\Rightarrow \theta = \arctan\left(\frac{b}{a}\right)$

First instinct!

$\theta = \arctan(x)$

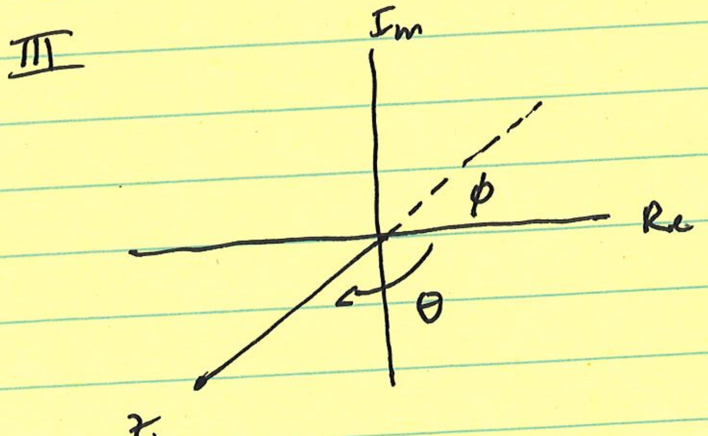






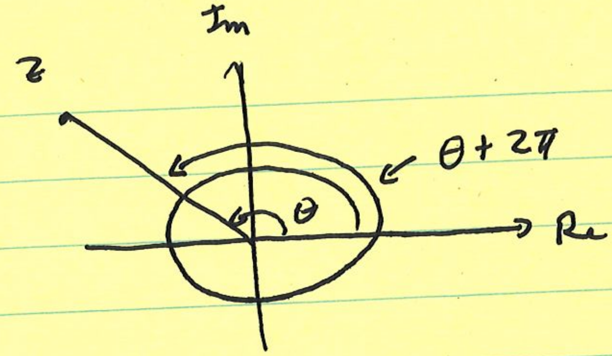
$\phi = \arctan\left(\frac{b}{a}\right)$

$\theta = \phi + \pi$



$\phi = \arctan\left(\frac{b}{a}\right)$

$\theta = \phi - \pi$



If θ is a defining angle then so is

$\theta + 2\pi$

and $\theta + 4\pi$

and $\theta + 6\pi$

$\vdots$

If θ_0 is a value of $\arg z$, then so is $\theta_0 \pm 2\pi k$ for $k = 0, \pm 1, \pm 2, \ldots$

Complex exponential

z_1, z_2 $\quad z_1 \cdot z_2 = ?$

$$\frac{z_1}{z_2} = \frac{z_1 \cdot \bar{z}_2}{z_2 \bar{z}_2} = \frac{}{|z_2|^2}$$

$z^{100} = ?$, $z^{1/5} = (a+ib)^{1/5}$

$\neq a^{1/5} + i b^{1/5}$ wrong!

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

Exponentials have nice properties

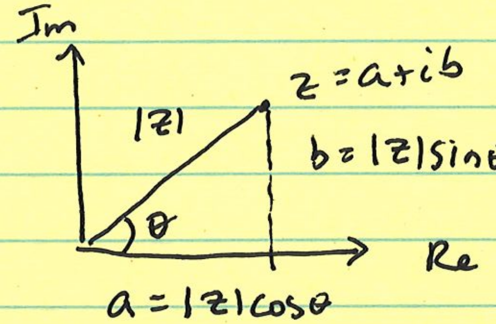
Given e^{ax} $\quad \frac{d}{dx} e^{ax} = a e^{ax}$

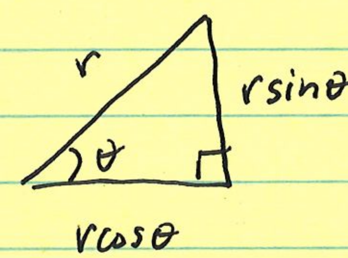
$$e^{i\theta_1} \cdot e^{i\theta_2} = e^{i(\theta_1 + \theta_2)}$$

$$\frac{e^{i\theta_1}}{e^{i\theta_2}} = e^{i(\theta_1 - \theta_2)}$$

$$(re^{i\theta})^{100} = r^{100} e^{i\theta(100)} \qquad z = |z| e^{i(\arg z)}$$

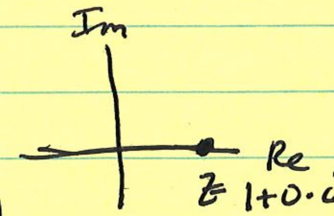
$$(re^{i\theta})^{1/5} = r^{1/5} e^{i\theta/5}$$





$$\begin{aligned} z = a + ib &= |z|\cos\theta + i\,|z|\sin\theta \\ &= |z|\,(\underbrace{\cos\theta + i\sin\theta}_{e^{i\theta}}) \\ &= |z|\,e^{i\theta} \qquad \theta = \arg z \\ &= |z|\,e^{i(\theta+2\pi)} \\ &= |z|\,e^{i(\theta+4\pi)} \\ &\ \vdots \\ &= |z|\,e^{i(\theta+2\pi k)} \qquad k = 0, \pm 1, \pm 2, \dots \end{aligned}$$

Another way to represent 1



$$\begin{aligned} 1 &= \cos(0 + 2\pi k) + i\sin(0 + 2\pi k) \\ &= 1 \cdot e^{i(2\pi k)} \end{aligned}$$

$$x^3 = 1 \Rightarrow x = 1^{1/3} \Rightarrow x = (e^{i2\pi k})^{1/3} = e^{i2\pi k/3}$$