Lecture 3: Wrap up complex numbers,
Introduction to systems of linear equations

Review: Last lecture we defined the complex number of the form $z = a + ib$, which has the defining properties:

- Modulus/absolute value: $|z| = \sqrt{a^2 + b^2}$
- $z\bar{z} = |z|^2 = a^2 + b^2$

and argument/phase: $\theta = 2\pi k, \quad k = 0, 1, ...$

where $\theta$ was determined by the quadrant

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- $I$: $\theta = \arctan \left( \frac{b}{a} \right)$
- $II$: $\theta = \arctan \left( \frac{b}{a} \right) + \pi$
- $III$: $\theta = \arctan \left( \frac{b}{a} \right) - \pi$

TopHat question was how many solutions are there to $x^3 = 1$

Another way to represent 1 is

$1 = \cos \left( 0 + 2\pi k \right) + i \sin \left( 0 + 2\pi k \right) \quad k = 0, 1, 2, ...$

$Z = re^{i\theta}$

$r = |Z|$

Recall, $re^{i\theta} = r(\cos \theta + is\sin \theta)$

$1 = e^{i(2\pi k)}$
$x^3 = 1 \Rightarrow x = \frac{1}{3} \Rightarrow x = \left( e^{\frac{2\pi i k}{3}} \right)^{1/3} = e^{\frac{2\pi i k}{3}}$

$k = 0 \Rightarrow 1$

$k = 1 \Rightarrow x = \cos \left( \frac{2\pi}{3} \right) + i \sin \left( \frac{2\pi}{3} \right)$

$k = 2 \Rightarrow x = \cos \left( \frac{4\pi}{3} \right) + i \sin \left( \frac{4\pi}{3} \right)$

Remark: mth roots of an arbitrary complex number \( z = r e^{i\theta} \) are the m distinct roots of \( z \) are given by

\[ z^{1/m} = m^{1/2} \left( \cos \left( \frac{\theta + 2\pi k}{m} \right) + i \sin \left( \frac{\theta + 2\pi k}{m} \right) \right) \]

\[ k = 0, 1, 2, \ldots, m-1 \]
Systems of linear equations

Systems of linear equations play an important and motivating role in the subject of linear algebra.

Basic definition:
A **linear equation** in unknowns $x_1, x_2, \ldots, x_n$ is an equation that can be put in the standard form

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$  \hspace{1cm} (1)

where $a_1, a_2, \ldots, a_n$ and $b$ are constants, termed coefficients.

A solution of (1) is a set of values $x_1, \ldots, x_n$ that satisfy (1).

Example: Consider the linear equation

$$\frac{1}{2}x_1 + x_2 = b$$

$$a_1 = \frac{1}{2}, \quad a_2 = 1, \quad b = 0$$

Rearrange terms to show

$$x_1 = -2x_2$$

infinite number of solutions,

choose any $x_2$ and solve for $x_1$

$x_2$ is a **free variable**
System of linear equations is a list of linear equations with the same unknowns.

A system of \( m \) linear equations \( L_1, L_2, \ldots, L_m \) in \( n \) unknowns \( x_1, x_2, \ldots, x_n \) can be put in the standard form
\[
L_i : a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n = b_i
\]
\[
L_j : a_{j1}x_1 + a_{j2}x_2 + \ldots + a_{jn}x_n = b_j
\]
\[
L_m : a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m
\]

\( b_i, a_{ij} \) : constants  \( i \) : equation \( L_i \)
\( j \) : coefficient of unknown variable \( x_j \)

The system above is called an \( m \times n \) (read \( m \) by \( n \)) system. It is called a square system if \( m=n \), that is, the number \( m \) of equations is equal to the number \( n \) of unknowns.

Consider the following \( 2 \times 2 \) system
\[
L_1 : \frac{1}{2}x_1 + x_2 = 0
\]
\[
L_2 : 0 \cdot x_1 + x_2 = 5
\]

\[
L_2 : x_2 = 5
\]
\[
L_1 : x_1 = -2x_2 = -2 \cdot 5 = -10
\]

One unique solution
\[
x_1 = -10, \ x_2 = 5
\]
Example

Now consider the case
\[ L_1 : \frac{1}{2}x_1 + x_2 = 0 \]
\[ L_2 : 0 \cdot x_1 + 0 \cdot x_2 = b \]

if \( b \neq 0 \), the \( L_2 \) is a degenerate linear equation (inconsistent) and the 2x2 system has no solution.

if \( b = 0 \), the 2x2 system has infinite solutions.

Finally, consider the case
\[ L_1 : \frac{1}{2}x_1 + x_2 = 0 \]
\[ L_2 : x_1 + 2x_2 = 0 \]

\[ L_1 \Rightarrow x_2 = -2x_2 \quad \text{some!} \]
\[ L_2 \Rightarrow x_1 = -2x_2 \]

Summary: A system of linear equations has three types of solutions

1. Exactly one solution
2. No solution
3. Infinite solutions

We can visualize these 3 scenarios plotting the equations
(1) One solution
\[ L_1: \frac{1}{2}x_1 + x_2 = 0 \Rightarrow x_2 = -\frac{1}{2}x_1 \]
\[ L_2: x_2 = 5 \]
\[ x_2 = 5 \]
\[ x_1 = -10 \]

(2) No solution
\[ L_1: \frac{1}{2}x_1 + x_2 = 0 \Rightarrow x_1 = -2x_2 \]
\[ L_2: \frac{1}{2}x_1 + x_2 = 1 \Rightarrow x_1 = -2x_2 + 2 \]

(3) Infinite solutions
\[ L_1: \frac{1}{2}x_1 + x_2 = 0 \]
\[ L_2 \Rightarrow x_1 + 2x_2 = 0 \]
Questions:
What are the properties that resulted in these types of solutions?
What criteria must be met to ensure existence of a unique solution?

Case (1): Exactly one solution

\[ L_2 \quad \text{This system is called independent} \]

Linearly independent system
A system of \( m \) linear equations is linearly independent if no equation can be written as a linear combination of the other.

Case (2): Infinite solutions
We had a linearly dependent system
\[ L_1: \frac{1}{2}x_1 + x_2 = 0 \]
\[ L_2: x_1 + 2x_2 = 0 \]

Note that \( L_2 \) is just \( L_1 \) multiplied by 2.
Summary: A square \((n \times n)\) system has a unique solution if and only if the system is linearly independent and consistent.