Lecture 3 Wrap up complex numbers,
Introduction to systems of linear equations

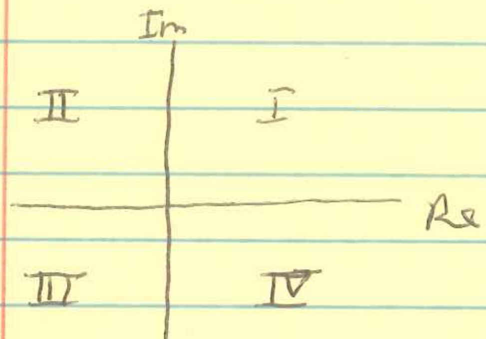
Review: Last lecture we defined the complex number of the form $z = a + ib$, which has the defining properties:

modulus / absolute value $|z| = \sqrt{a^2+b^2}$

$$z\bar{z} = |z|^2 = a^2+b^2$$

and argument / phase $\theta \pm 2\pi k$ $k = 0, 1, \ldots$

where θ was determined by the quadrant



I, IV : $\theta = \arctan\left(\frac{b}{a}\right)$

II : $\theta = \arctan\left(\frac{b}{a}\right) + \pi$

III : $\theta = \arctan\left(\frac{b}{a}\right) - \pi$

TopHat question was how many solutions are there to $x^3 = 1$

Another way to represent 1 is

$1 = \cos(0 + 2\pi k) + i\sin(0 + 2\pi k)$ $k = \pm 1, \pm 2, \ldots$ $k = 0$

$z = re^{i\theta}$

$r = |z|$

Recall, $re^{i\theta} = r(\cos\theta + i\sin\theta)$

$$1 = e^{i(2\pi k)}$$

$$x^3 = 1 \Rightarrow x = 1^{1/3} \Rightarrow x = \left(e^{i2\pi k}\right)^{1/3}$$
$$= e^{i2\pi k/3}$$

$k = 0 \Rightarrow 1$

$k = 1 \Rightarrow x = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)$

$k = 2 \Rightarrow x = \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)$

Remark: mth roots of an arbitrary complex number $z = re^{i\theta}$ are the m distinct m-th roots of z are given by

$$z^{1/m} = \sqrt[m]{|z|}\, e^{i(\theta + 2\pi k)/m}$$

$k = 0, 1, 2, \ldots, m-1$

Systems of linear equations

Systems of linear equations play an important and motivating role in the subject of linear algebra

Basic definition:

A linear equation in unknowns $x_1, x_2, \dots, x_n$ is an equation that can be put in the standard form

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = b \qquad (1)$$

where $a_1, a_2, \dots, a_n$ and b are constants, termed coefficients

A solution of (1) is a set of values $x_1, \dots, x_n$ that satisfy (1).

Example: Consider the linear equation

$$\frac{1}{2}x_1 + x_2 = 0$$

$$a_1 = \frac{1}{2},\ a_2 = 1,\ b = 0$$

Rearrange terms to show

$x_1 = -2x_2 \Rightarrow$ infinite number of solutions

choose any x_2 and solve for x_1

x_2 is a free variable

System of linear equations is a list of linear equations with the same unknowns.

A system of m linear equations $L_1, L_2, \ldots, L_m$ in n unknowns $x_1, x_2, \ldots, x_n$ can be put in the standard form

$$L_1: a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$
$$L_2: a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$
$$\vdots$$
$$L_m: a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m$$

b_i, a_{ij} : constants

i: refers to equation L_i
j: coefficient of unknown variable x_j

The system above is called an $m \times n$ (read m by n) system. It is called a square system if $m = n$, that is, the number m of equations is equal to the number n of unknowns.

Consider the following 2×2 system

$$L_1: \tfrac{1}{2}x_1 + x_2 = 0$$
$$L_2: 0 \cdot x_1 + x_2 = 5$$

$L_2: x_2 = 5$

$L_1: x_1 = -2x_2 = -2 \cdot 5 = -10$

one unique solution

$x_1 = -10, \ x_2 = 5$

Example

Now consider the case

$L_1: \frac{1}{2}x_1 + x_2 = 0$

$L_2: 0 \cdot x_1 + 0 \cdot x_2 = b$

if $b \neq 0$, the L_2 is a degenerate linear equation (inconsistent) and the 2×2 system has no solution.

if $b = 0$, the 2×2 system has infinite solutions.

Finally, consider the case

$L_1: \frac{1}{2}x_1 + x_2 = 0$

$L_2: x_1 + 2x_2 = 0$

$L_1 \Rightarrow x_1 = -2x_2$ same!

$L_2 \Rightarrow x_1 = -2x_2$

Summary: A system of linear equations has three types of solutions

(1) Exactly one solution

(2) No solution

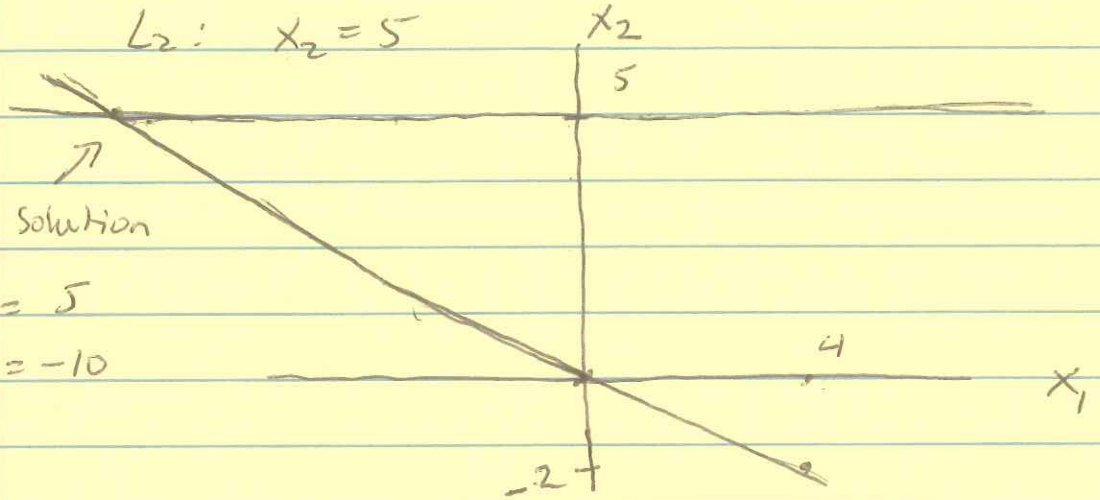
(3) Infinite solutions

We can visualize these 3 scenarios plotting the equations

(1) One solution

$L_1: \frac{1}{2}x_1 + x_2 = 0 \Rightarrow x_2 = -\frac{1}{2}x_1$

$L_2: x_2 = 5$



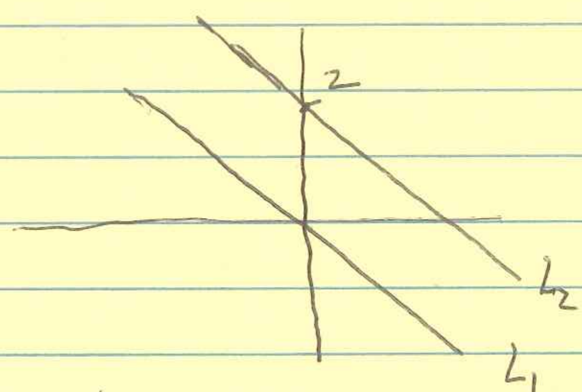
$x_2 = 5$

$x_1 = -10$

(2) No solution

$L_1 : \frac{1}{2}x_1 + x_2 = 0 \Rightarrow x_1 = -2x_2$

$L_2 = \frac{1}{2}x_1 + x_2 = 1 \Rightarrow x_1 = -2x_2 + 2$

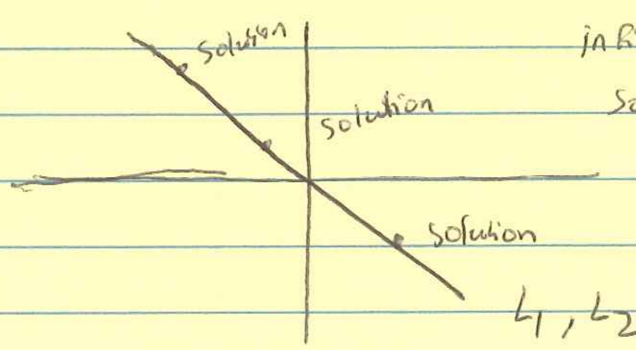


no intersect/
no solution

(3) Infinite solutions

$L_1 : \frac{1}{2}x_1 + x_2 = 0$

$L_2 : x_1 + 2x_2 = 0$



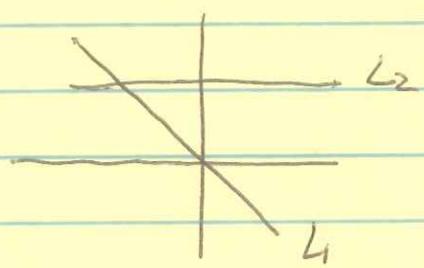
infinite
Solutions

Questions:

What are the properties that resulted in these types of solutions?

What criteria must be met to ensure existence of a unique solution?

Case (1) : Exactly one solution



This system is called independent

Linearly independent System

A system of m linear equations is linearly independent if no equation can be written as a linear combination of the others

Case (3) : Infinite solutions

We had a linearly dependent system

$L_1: \frac{1}{2}x_1 + x_2 = 0$

$L_2: x_1 + 2x_2 = 0$

Note that L_2 is just L_1 multiplied by 2

<u>Summary</u>: A square ($n \times n$) system has a unique solution if and only if the system is linearly independent and consistent.