Continued from Lecture 2

$$x^3 = 1 \Rightarrow (x^3)^{1/3} = 1^{1/3} = x = 1^{1/3}$$

$$1 = \underbrace{\cos(0 + 2\pi k) + i\sin(0 + 2\pi k)}$$

$$\cancel{\sqrt{a^2+b^2} = \sqrt{1^2+1^2} = \sqrt{2}}$$

$$re^{i\theta} = r(\cos\theta + i\sin\theta)$$
$$= r\cdot\overset{1}{\cos\theta} + i\, r\cdot\overset{0}{\sin\theta} \qquad \theta = 0 + 2\pi k$$
$$= 1 \qquad \text{Let } r = 1$$

$$1 = e^{i(2\pi k)}$$

$$x = \left(e^{i2\pi k}\right)^{1/3} = e^{i2\pi k/3}$$

$k = 0 \Rightarrow e^0 \Rightarrow 1$

$k = 1 \Rightarrow e^{i2\pi/3} \Rightarrow x = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)$

$k = 2 \Rightarrow e^{i4\pi/3} \Rightarrow x = \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)$

$k = 3 \Rightarrow x = \overset{1}{\cos}(2\pi) + i\,\overset{0}{\sin(2\pi)}$

$k = 4 \Rightarrow$ same solution from $k = 1$

Remark: Given an arbitrary complex number $z = re^{i\theta}$, the m distinct m-th roots of z are given by

$$z^{1/m} = \sqrt[m]{|z|}\, e^{i(\theta + 2\pi k)/m}$$
$$k = 0, 1, 2, \ldots, m-1$$

Lecture 3

Systems of linear equations

Linear equation

$$a_1 x_1 + a_2 x_2 + \dots + a_n x_n = b \quad (1)$$

unknowns : $x_1, x_2, \dots x_n$

$a_1, a_2, \dots, a_n$ and b are constants, termed coefficients

A solution of (1) is a set of values $x_1, \dots, x_n$ that satisfy (1).

TH Q1 Consider the linear equation

$$\frac{1}{2}x_1 + x_2 = 0$$

$$a_1 = \frac{1}{2}, \; a_2 = 1, \; b = 0$$

$$x_1 = -2x_2$$

$\Rightarrow$ Infinite number of solutions. Choose any x_2 and solve for x_1

x_2 is a <u>free variable</u>

System of linear equations is a list of linear equations with the same unknowns

$$L_1 : a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n = b_1$$
$$L_2 : a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n = b_2$$
$$\vdots$$
$$L_m : a_{m1} x_1 + a_{m2} x_2 + \dots + a_{mn} x_n = b_m$$

b_i, a_{ij} : constants

i : refers to equation L_i

j : coefficient of unknown variable x_j

The system above is called an $m \times n$ (read m by n) system. It is called a square system if $m = n$. That is, the number m of equations is equal to the number of unknowns.

TH Q2 $\quad$ 2×2 system

$$L_1 : \tfrac{1}{2} x_1 + x_2 = 0$$
$$L_2 : 0 \cdot x_1 + x_2 = 5$$

L_2 : $x_2 = 5$

L_1 : $x_1 = -2x_2 = -2 \cdot 5 = -10$

one unique solution

$x_1 = -10$, $x_2 = 5$

TH Q 3

$$L_1: \tfrac{1}{2}x_1 + x_2 = 0$$
$$L_2: 0\cdot x_1 + 0\cdot x_2 = b$$

if $b \neq 0$, then L_2 is a degenerate linear equation (inconsistent) and the 2×2 system has no solution.

if $b = 0$, then the 2×2 system has infinite solutions.

TH Q4

$$L_1: \tfrac{1}{2}x_1 + x_2 = 0$$
$$L_2: x_1 + 2x_2 = 0$$

$L_1: \Rightarrow \quad x_1 = -2x_2$ same!

$L_2 \Rightarrow \quad x_1 = -2x_2$

Summary: A system of linear equations has three types of solutions

(1) Exactly one solution

(2) No solution

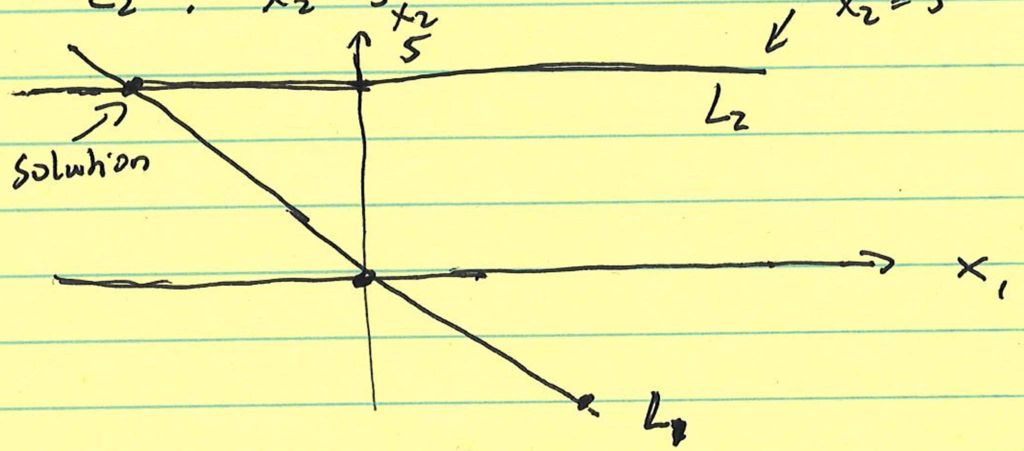
(3) Infinite solutions

We can visualize these 3 scenarios ~~by~~ plotting the equations

(1) One solution

$L_1 : \frac{1}{2}x_1 + x_2 = 0$

$L_2 : x_2 = 5$

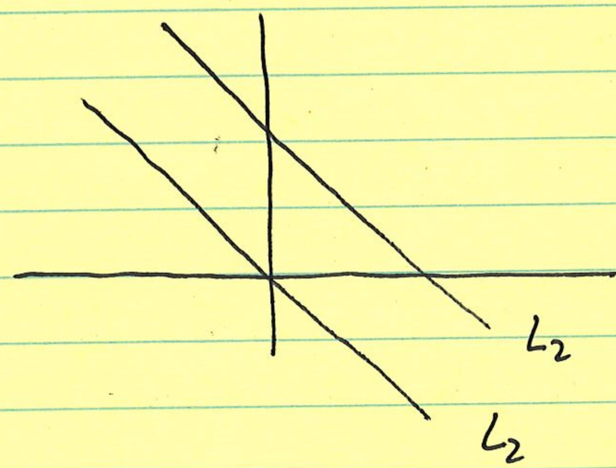


(2) No solution

$L_1 : \frac{1}{2}x_1 + x_2 = 0 \Rightarrow x_1 = -2x_2$

$L_2 : \frac{1}{2}x_1 + x_2 = 1 \Rightarrow x_1 = -2x_2 + 2$

$\Rightarrow 2 \neq 0$

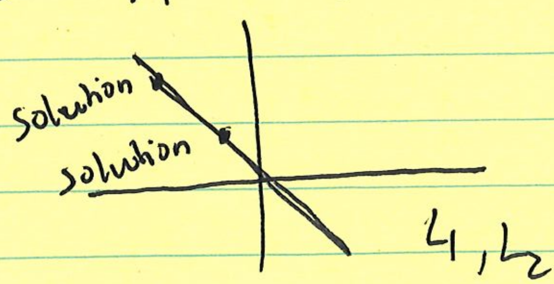


no intersect/
no solution

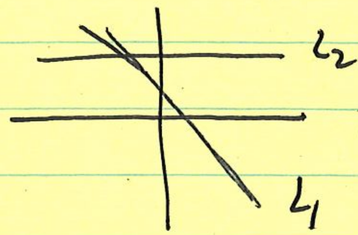
(3) Infinite solutions

$L_1 : \frac{1}{2}x_1 + x_2 = 0$

$L_2 : x_1 + 2x_2 = 0$



Case (1) : Exactly one solution



This system is called independent

Linearly independent system

A system of m linear equations is linearly independent if no equations can be written as a linear combination of the others.

Case (3) : Infinite solutions

We had a linearly dependent system

$$L_1 = \tfrac{1}{2}x_1 + x_2 = 0$$
$$L_2 = x_1 + 2x_2 = 0$$

Note that L_2 is just L_1 multiplied by 2

Summary : A square ~~[illegible]~~ $(m \times m)$ system has a unique solution if and only if the system is linearly independent and consistent.

Elementary Operations

Example system

$$L_1: x_1 + x_2 + 5x_3 = 6$$
$$L_2: x_1 + 2x_2 + x_3 = 1$$
$$L_3: 2x_1 + 10x_2 + 2x_3 = 10$$

[E1] Interchange two of the equations denoted by $L_i \Leftrightarrow L_j$ "Interchange L_i and L_j".

$L_1 \Leftrightarrow L_3 \Rightarrow$
$$L_1: 2x_1 + 10x_2 + 2x_3 = 10$$
$$L_2: x_1 + 2x_2 + x_3 = 1$$
$$L_3: x_1 + x_2 + 5x_3 = 6$$

[E2] Replace an equation by a non-zero multiple of itself denoted by $kL_i \rightarrow L_i$ "Replace L_i by kL_i"

$2L_1 \rightarrow L_1 \Rightarrow$
$$L_1: 4x_1 + 20x_2 + 4x_3 = 20$$
$$L_2: x_1 + 2x_2 + x_3 = 1$$
$$L_3: x_1 + x_2 + 5x_3 = 6$$