4  Week 2: 1/17/17

4.1 Summary

Last week we introduced separation of variables which led to S-L eigenvalue problems. Thus far the type of eigenvalue problems we looked at had solutions corresponding to Fourier series expansion. In the case of homogeneous boundary conditions we were left with a sine series expansion. When we looked at periodic boundary conditions we had to retain the complete Fourier series expansion.

4.2 Revisit S-L problem

The prescribed homogeneous boundary conditions

\[ \beta_1 \phi(a) + \beta_2 \frac{d\phi}{dx}(a) = 0 \]  
(97)

\[ \beta_3 \phi(b) + \beta_4 \frac{d\phi}{dx}(b) = 0 \]  
(98)

\( \beta_1, \ldots, \beta_4 \) are real numbers for the S-L problem pertain to a subset of eigenvalue problems termed "regular" and add the condition that all eigenvalues are real and there exists a unique eigenfunction pertaining to each eigenvalue.

Last week we looked at a problem with the following periodic boundary conditions (periodic S-L) that arose from polar coordinates:

\[ \phi_{xx} + \lambda \phi = 0 \]

\[ \phi(-\pi) = \phi(\pi) \]

\[ \phi_x(-\pi) = \phi_x(\pi) \]

Note that we obtained a complete biorthogonal system instead of simply orthogonal. Each eigenvalue \( \lambda_n \) had two eigenfunctions \( \{\cos(n\pi), \sin(n\pi)\} \) (note: no more that two allowed for a second order system). Also note that \( \lambda = 0 \) has only one eigenfunction (constant). It still holds that all eigenvalues are real.

4.2.1 Self-adjoint operator

We can write the S-L problem as an operator. Let \( L \) be the linear operator

\[ L(y) \equiv \frac{d}{dx} \left[ p(x) \frac{dy}{dx} \right] + q(x)y \]
	hen then the S-L problem can be written as

\[ L(\phi) + \lambda \sigma(x) \phi = 0 \]

. Consider applying the operator to two different variable \( u \) and \( v \), we then calculate

\[ uL(v) - vL(u) \]

and manipulating the RHS we get

\[ uL(v) - vL(u) = \frac{d}{dx} \left[ p \left( u \frac{dv}{dx} - v \frac{du}{dx} \right) \right] \]
This is known as Lagrange’s identity. The integral form
\[
\int_a^b [uL(v) - vL(u)] = p\left( u \frac{dv}{dx} - v \frac{du}{dx} \right)|_a^b
\]
is also known as Green’s formula.

If \( u \) and \( v \) satisfy the same set of boundary conditions in the regular S-L problem then one can show that the RHS vanishes and it holds that
\[
\int_a^b [uL(v) - vL(u)] = 0
\]
which by definition means that the linear operator \( L \) is self-adjoint. This also holds for periodic boundary conditions.

Now if we apply this operator to an eigenfunction we get
\[
L(\phi_n) = -\lambda_n \sigma(x) \phi_n
\]
much like in linear algebra when \( Av = \lambda v \). So if we take the functions \( u \) and \( v \) to be eigenfunctions \( \phi_m \) and \( \phi_n \), respectively, we get
\[
\int_a^b [\phi_m L(\phi_n) - \phi_n L(\phi_m)] = 0 \tag{99}
\]
\[
(\lambda_m - \lambda_n) \int_a^b \phi_n \phi_m \sigma dx = 0 \implies \int_a^b \phi_n \phi_m \sigma dx = 0 \text{ for } \lambda_n \neq \lambda_m \tag{100}
\]
Thus we have the eigenfunctions corresponding to different eigenvalues are orthogonal.

Now we will show that they are real. Consider the complex conjugate of the S-L equation. So we have
\[
\bar{L}(\phi) + \bar{\lambda} \sigma \phi = 0 \tag{101}
\]
\[
L(\bar{\phi}) + \bar{\lambda} \sigma \bar{\phi} = 0 \tag{102}
\]
We plug this into the the LHS of Green’s formula and get
\[
(\lambda - \bar{\lambda}) \int_a^b \phi \bar{\phi} \sigma dx = 0
\]
Since \( \phi \bar{\phi} = |\phi|^2 \geq 0 \), the inegrand is positive definite so the equality can only hold if \( \lambda = \bar{\lambda} \).

Note: can use the RHS of Lagrange’s identity to show the eigenfunctions are unique for the regular S-L problem.

### 4.3 Rayleigh quotient

Rayleigh quotient is an analytical expression for the eigenvalue. To derive the Rayleigh quotient take the S-L problem,
\[
\frac{d}{dx} \left[ p(x) \frac{d\phi}{dx} \right] + q(x) \phi + \lambda \sigma(x) \phi = 0
\]
multiply by \( \phi \) and integrate over \( a \) to \( b \) and solve for \( \lambda \).
\[
\lambda = -\int_a^b \left[ \phi \frac{d}{dx} \left( p \frac{d\phi}{dx} \right) + q \phi^2 \right] dx \int_a^b \phi^2 \sigma dx \tag{103}
\]
integrate by parts and we arrive at
\[
\lambda = \frac{-p\phi \frac{d\phi}{dx} \bigg|_a^b - \int_a^b \left[ p \left( \frac{d\phi}{dx} \right)^2 + q\phi^2 \right] dx}{\int_a^b \phi^2 \sigma dx}
\] (104)

Can give us some useful information even if we can’t evaluate the expression.
Consider
\[
\phi_{xx} + \lambda \phi = 0
\] (105)
\[
\phi(0) = 0
\] (106)
\[
\phi(L) = 0
\] (107)

The Rayleigh coefficient reduces to
\[
\lambda = \frac{\int_0^L (d\phi/dx)^2 dx}{\int_0^L \phi^2 dx}
\]
we see that both the numerator and denominator are positive \( \lambda \geq 0 \) This helps us to avoid having to check all the cases of possible \( \lambda \)'s by providing constraints.

### 4.4 Different solution arising from S-L problems

#### 4.4.1 Bessel

Bessel functions the solutions \( y(x) \) of Bessel’s differential equation
\[
x^2 y_{xx} + xy_x + (x^2 - m^2)y = 0
\]
The general solution is
\[
f = c_1 J_m(x) + c_2 Y_m(x)
\]
\( J_m \) is the Bessel function of the first kind of order \( m \). \( Y_m \) called the Bessel function of the second kind of order \( m \). \( Y_m \) is not bounded and so bounded conditions \( \Rightarrow c_2 = 0 \)
\( J_m \) looks like decaying oscillation

Example: vibrating membrane
\[
U_{tt} = c^2 \nabla^2 U
\] (108)
\[
U_{tt} = c^2 \left( \frac{1}{r} U_r + U_{rr} + \frac{1}{r^2} U_{\theta\theta} \right)
\] (109)

We assume a solution of the form
\[
U(r, \theta, t) = R(r)H(\theta)T(t)
\]
plugging into the ODE we get
\[
\frac{1}{c^2} \frac{T''}{T} = \frac{1}{r} \frac{R'}{R} + \frac{R''}{R} + \frac{1}{r^2} \frac{H''}{H} = -\lambda^2
\]
So for the time-dependent portion we get
\[
T'' + c^2 \lambda^2 T = 0
\]
which we have seen before and already know the solution is sinusoidal in time. In the second system we have

\[
\frac{R'}{r} + r^2 \frac{R''}{R} + \frac{H''}{H} = -\lambda^2 r^2
\]

We know \( H \) has periodic boundary conditions

\[
\frac{H''}{H} = -n^2
\]

So we have

\[
H = A \cos(n\theta) + B \sin(n\theta)
\]

and using this we can find an ODE for \( r \)

\[
rR' + r^2 R'' - Rn^2 + \lambda^2 r^2 R = 0
\]

\[
r^2 R'' + R'r + R(\lambda^2 r^2 - n^2) = 0
\]

\[
\frac{d}{dr}(rR') - \frac{n^2}{r} R + \lambda^2 r R = 0
\]

we have initial conditions \( R(a) = 0 \), \( R(r) \) for \( 0 \leq r \leq a \), and \( R(0) < \infty \) make the substitution \( x = \lambda r \) and \( y(\lambda r) = R(r) \) then we have

\[
\lambda y(\lambda r) = R'(r)
\]

and

\[
\lambda^2 y''(\lambda r) = R''(r)
\]

If we plug into the second equation

\[
\lambda^2 r^2 y'' + \lambda ry' + y(\lambda^2 r^2 - n^2) = 0
\]

or

\[
x^2 y'' + xy' + y(x^2 - n^2) = 0
\]

which is a Bessel equation. We have \( R(a) = 0 \) and \( R(0) \) is bounded (around \( r = 0 \)) The general solution is made up of two part (first and second kind)

\[
y(x) = \alpha J_n(x) + \beta \tilde{J}_n(x)
\]

the first term is bounded and is the Bessel function and the second term \( (\text{Neuman}) \) is unbounded at \( x = 0 \) which \( \implies \beta = 0 \).

\[
y(x) = \alpha J_n(x)
\]

applying the initial condition we get

\[
0 = R(a) = \alpha J_n(\lambda a)
\]

recall \( y(\lambda r) = R(r) \)

We need to choose \( \lambda \) s.t. \( J_n(\lambda a) = 0 \).

\[
\implies a\lambda = z_{mn} \text{ mth zero of function } J_n
\]
Putting everything together

\[ U(r, \theta, t) = R(r)H(\theta)T(t) \]

\[ = J_n \left( \frac{Z_{m,n}}{a} r \right) \left( A \cos(n\theta) + B \sin(n\theta) \right) \left( C \cos \left( \frac{c Z_{m,n}}{a} t \right) + D \sin \left( \frac{c Z_{m,n}}{a} t \right) \right) \]  \hspace{0.5cm} (110)

\[ = J_n \left( \frac{Z_{m,n}}{a} r \right) \left( A \cos(n\theta) + B \sin(n\theta) \right) \left( C \cos \left( \frac{c Z_{m,n}}{a} t \right) + D \sin \left( \frac{c Z_{m,n}}{a} t \right) \right) \]  \hspace{0.5cm} (111)

recall for \( T \) we had \( T'' + c^2 \lambda^2 T = 0 \)...

For each fixed \( n \)

\[ J_n \left( \frac{Z_{m,n}}{a} r \right) \]

is a complete system and

\[ \int_0^a J_n \left( \frac{Z_{m,n}}{a} r \right) J_n \left( \frac{Z_{p,n}}{a} r \right) r dr = \begin{cases} 0 & p \neq m \\ \frac{a^2}{2} [J_{n+1}(Z_{m,n})]^2 & m = p \end{cases} \]

4.4.2 Finite difference scheme

\[ U_t = DU_{xx} \]

finite difference approximation for the second derivative with a \( O(\Delta x)^2 \) truncation error and \( O(\Delta x) \) truncation error for first derivative approximation in time

\[ x_j = j \Delta x \]
\[ t_m = m \Delta t \]

\[ u_j^{m+1} = u_j^m + \frac{s(u_{j+1}^m - 2u_j^m + u_{j-1}^m)}{\Delta t} \]

where \( s = k(\Delta t)/(\Delta x)^2 \)

\[ U_t = DU_{xx} + Q(U) \]

We can just discretize in space

\[ \kappa \equiv D/(\Delta x)^2 \]

\[ du_j/dt = \kappa(u_{j+1} - 2u_j + u_{j-1}) + Q(u_j) \]

\[ j \in [1, n-1] \] where \( n = L/\Delta x \)

discuss 2D system