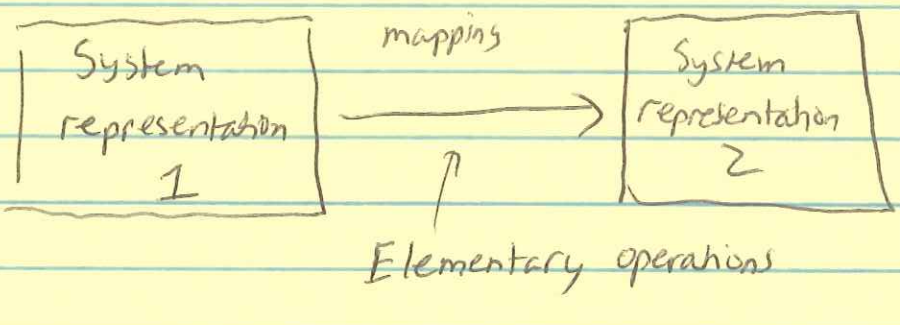
Lecture 4 Gaussian Elimination

Last lecture we explored types of solutions that emerge in a system of linear equations. We noted that <u>linear dependence</u> of equations play a <u>key role</u> in the existence of unique solutions.

That is, can a linear equation be written as a linear combination of the remaining in the system.

This relationship can be easily determined by mapping the system to an equivalent simplified system in which linear dependence is made obvious.



<u>Elementary Operations</u> 3 operations

Example system

$$L_1: \ x_1 + x_2 + 5x_3 = 6$$

$$L_2: \ x_1 + 2x_2 + x_3 = 1$$

$$L_3: \ 2x_1 + 10x_2 + 2x_3 = 10$$

[E1] Interchange two of the equations denoted by $L_i \leftrightarrow L_j$ "Interchange L_i and L_j"

$$L_1 \leftrightarrow L_3 \Rightarrow \begin{array}{l} L_1: 2x_1 + 10x_2 + 2x_3 = 10 \\ L_2: x_1 + 2x_2 + x_3 = 1 \\ L_3: x_1 + x_2 + 5x_3 = 6 \end{array}$$

[E2] Replace an equation by a non-zero multiple of itself denoted by

$kL_i \rightarrow L_i$ "Replace L_i by kL_i"

$$2L_1 \rightarrow L_1 \qquad \begin{array}{l} L_1: 2x_1 + 2x_2 + 10x_3 = 12 \\ L_2: x_1 + 2x_2 + x_3 = 1 \\ L_3: 2x_1 + 10x_2 + 2x_3 = 10 \end{array}$$

[E3] Replace an equation by ~~an non-zero multiple~~ the sum of a multiple of another equation and itself denoted by

$kL_i + L_j \rightarrow L_j$ "Replace L_j by $kL_i + L_j$"

$$2L_1 + L_2 \rightarrow L_2 \qquad \begin{array}{l} L_1: x_1 + x_2 + 5x_3 = 6 \\ L_2: 2(x_1 + x_2 + 5x_3 = 6) + (x_1 + 2x_2 + x_3 = 1) \\ L_3: 2x_1 + 10x_2 + 2x_3 = 10 \end{array}$$

$$\Rightarrow \begin{array}{l} x_1 + x_2 + 5x_3 = 6 \\ 3x_1 + 4x_2 + 11x_3 = 13 \\ 2x_1 + 10x_2 + 2x_3 = 10 \end{array}$$

Theorem: Suppose a system $\mathcal{M}$ of linear equations is obtained from a system $\mathcal{L}$ of linear equations by a finite sequence of elementary operations. Then $\mathcal{M}$ and $\mathcal{L}$ have the same solutions

[E_1] $L_i \leftrightarrow L_j$

[E_2] $kL_i \rightarrow L_i$

[E_3] $kL_i + L_j \rightarrow L_j$

Goal: Use elementary row operations to put the system in an equivalent form that is simpler to solve

Consequence: Simpler structure tells us how many solutions there are.

What are these simpler forms??

Triangular Form:

$$\begin{array}{llll}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 = b_1 & (1) \\
0 \cdot x_1 \quad a_{22}x_2 + a_{23}x_3 + a_{24}x_4 = b_2 & (2) \\
0 \cdot x_1 + 0 \cdot x_2 \quad a_{33}x_3 + a_{34}x_4 = b_3 & (3) \\
0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 \quad a_{44}x_4 = b_4 & (4)
\end{array}$$

Easy to back solve for x_4, x_3, x_2, x_1 through back substitution

Important note: A system can be put in triangular form if and only if it has a unique solution!

echelon form:

$$\underset{\underset{\text{pivot variables}}{\nearrow}}{a_{11}x_1} + \underset{\underset{\text{free variables}}{\nearrow\!\!\nearrow}}{a_{12}x_2} + a_{13}x_3 + a_{14}x_4 = b_1$$

$$\underset{\underset{\text{pivot variable}}{\uparrow}}{a_{23}x_3} + a_{24}x_4 = b_2$$

$$\underset{\underset{\text{pivot variable}}{\uparrow}}{a_{34}x_4} = b_3$$

Remarks:

- Triangular form is always in echelon form
- echelon form is not always in triangular form

Theorem: consider a system of linear equations $(L_1, L_1, \ldots, L_m)$ in echelon form, say with m equations and n unknowns $(x_1, x_2, \ldots, x_n)$. There are two cases

(i) $m = n$ That is there are as many equations as unknowns (triangular form). Then the system has a unique solution

(ii) $m < n$ That is, there are more unknowns than equations. Then we can arbitrarily assign values to the $(n-m)$ free variables and solve uniquely for the m pivot variables, obtaining a solution of the system (infinite solutions)

Example of use of elementary operations to
arrive at echelon form

Example 1 $L_1 \quad \frac{1}{2}x_1 + x_2 = 0$

$L_2 \quad x_1 + 2x_2 = 0$

Apply
[E2] $2L_1 \rightarrow L_1$ $\Rightarrow$ $L_1 : x_1 + 2x_2 = 0$
$L_2 : x_1 + 2x_2 = 0$

Apply
[E3] $(-1)L_1 + L_2 \rightarrow L_2$

$$\underbrace{\begin{aligned} L_1 &: x_1 + 2x_2 = 0 \\ L_2 &: 0 + 0 = 0 \end{aligned}}_{\text{Echelon form}}$$

$m = 1$ one equation L_1

$n = 2$ unknown x_1, x_2

$m < n \Rightarrow (n-m)$ of $(2-1)$ free
variables and infinite Solutions

agrees with Tuesday's results

* consistent, not independent

Example 2

$L_1: \frac{1}{2}x_1 + x_2 = 0$

$L_2: \frac{1}{2}x_1 + x_2 = 1$

Apply [E2] $\Rightarrow$ $L_1: \frac{1}{2}x_1 + x_2 = 0$

$(-1)L_1 + L_2 \rightarrow L_2$ $\quad L_2: 0 + 0 = 1$

inconsistent

no solution

note: parallel equations are equivalent to a degenerate system

Example 3

$L_1: \frac{1}{2}x_1 + x_2 = 0$ Already in

$L_2: \quad x_2 = 5$ echelon form!

$m = 2$ equations L_1, L_2

$n = 2$ unknowns x_1, x_2

$m = n \Rightarrow$ unique solution

* consistent and independent

These systems were "easy" to solve since they were low dimension but how do we find echelon form of a general system?

There is algorithm!

Gaussian Elimination

Step by step reduction of the system yielding either a degenerate equation with no solution or an equivalent simpler system in triangular form or echelon form

(using elementary operations)

Example

$$\begin{aligned} 4x_1 + 2x_2 + x_4 &= 6 \\ x_1 + 2x_2 + x_3 + 2x_4 &= 1 \\ x_2 + x_4 &= 2 \\ x_1 + 3x_3 + 2x_4 &= 4 \end{aligned}$$

Arrange

$$\begin{aligned} L_1 \quad & 4x_1 + 2x_2 + 0\cdot x_3 + 1\cdot x_4 = 6 \\ L_2 \quad & x_1 + 2x_2 + 1\cdot x_3 + 2x_4 = 1 \\ L_3 \quad & 0\cdot x_1 + 1\cdot x_2 + 0\cdot x_3 + 1\cdot x_4 = 2 \\ L_4 \quad & 1\cdot x_1 + 0\cdot x_2 + 3x_3 + 2x_4 = 4 \end{aligned}$$

Put into the following form →

$$\left(\begin{array}{cccc|c} 4 & 2 & 0 & 1 & 6 \\ 1 & 2 & 1 & 2 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 1 & 0 & 3 & 2 & 4 \end{array}\right) \begin{array}{l} R_1 : \text{row } 1 \\ R_2 : \text{row } 2 \\ R_3 : \text{row } 3 \\ R_4 : \text{row } 4 \end{array}$$