Lecture 4

Echelon form

System is in echelon form if it has the following structure

Free variable

Example:
$$\begin{aligned} a_{11}x_1 &+ a_{12}x_2 + a_{13}x_3 + a_{14}x_4 = b_1 \\ 0\cdot x_1 &+ 0\cdot x_2 + a_{23}x_3 + a_{24}x_4 = b_2 \\ 0\cdot x_1 &+ 0\cdot x_2 + 0\cdot x_3 \quad a_{34}x_4 = b_3 \end{aligned}$$

pivot variables

The following is the general form of echelon

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots &+ a_{1n}x_n = b_1 \\ 0 \rightarrow a_{2j_2}x_{j_2} + a_{2j_{2+1}}x_{j_{2+1}} &+ \ldots + a_{2n}x_n = b_2 \\ \downarrow \qquad 0 \rightarrow \qquad &\vdots \\ \downarrow \qquad & \\ 0 \rightarrow a_{rj_r}x_{j_r} &+ \ldots + a_{rn}x_n = b_r \end{aligned}$$

pivot variables: $x_1, x_{j_2}, \ldots, x_{j_r}$

n unknowns, r equations

$r \leq n$

$1 < j_2 < \ldots < j_r$

Two cases than can occur (assuming system is consistent)

(1) $r = n$ # o' equations = # o' unknowns

$\Rightarrow$ system has a triangular form

Example:
$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 &= b_1 \\ 0 \quad a_{22}x_2 + a_{23}x_3 + a_{24}x_4 &= b_2 \\ 0 \quad a_{33}x_3 + a_{34}x_4 &= b_3 \\ 0 \quad a_{44}x_4 &= b_4 \end{aligned}$$

$\Rightarrow$ system has a unique solution

(2) $r < n$ more unknowns than equations
Assign arbitrary values to the $(n-r)$ free variables and solve uniquely for the r pivot variables

Return to Elementary Operations

[E_1] $L_i \leftrightarrow L_j$ Interchange two of the equations

"interchange L_i and L_j"

[E_2] $kL_i \to L_i$ Replace an equation by a non-zero multiple of itself

"Replace L_i by kL_i"

[E_3] $kL_i + L_j \to L_j$ Replace an equation by the sum of a multiple of another equation and itself

"Replace L_j by $kL_i + L_j$"

Example

$$
\begin{aligned}
L_1 &: x_1 + x_2 + 5x_3 = 6 \\
L_2 &: x_1 + 2x_2 + x_3 = 1 \\
L_3 &: 2x_1 + 10x_2 + 2x_3 = 10
\end{aligned}
$$

S1

Applied

[E1) $L_1 \leftrightarrow L_3$

[E2) $2L_1 \to L_1$

$$
\begin{aligned}
L_1 &: 4x_1 + 20x_2 + 4x_3 = 20 \\
L_2 &: x_1 + 2x_2 + x_3 = 1 \\
L_3 &: x_1 + x_2 + 5x_3 = 6
\end{aligned}
$$

[E3] $-1 L_2 + L_3 \to L_3$

$$L_1 : 4x_1 + 20x_2 + 4x_3 = 20$$
$$L_2 : x_1 + 2x_2 + x_3 = 1 \qquad S2$$
$$L_3 : 0 - x_2 + 4x_3 = 5$$

We have that S1 is equivalent to S2
$\Rightarrow$ they have the same solution

Back to Gaussian Elimination
Step by step reduction of the system yielding either degenerate (inconsistent) equation with no solution or an equivalent simpler system in triangular or echelon form

Steps demonstrated through example

$$4x_1 + 2x_2 + x_4 = 6$$
$$x_1 + 2x_2 + x_3 + 2x_4 = 1$$
$$x_2 + x_4 = 2$$
$$x_1 + 3x_3 + 2x_4 = 4$$

Arrange
$$L_1 : 4x_1 + 2x_2 + 0 \cdot x_3 + x_4 = 6$$
$$L_2 : x_1 + 2x_2 + x_3 + 2x_4 = 1$$
$$L_3 : 0 \cdot x_1 + x_2 + 0 \cdot x_3 + x_4 = 2$$
$$L_4 : x_1 + 0x_2 + 3x_3 + 2x_4 = 4$$

put in the following form

$$\left(\begin{array}{cccc|c} 4 & 2 & 0 & 1 & 6 \\ 1 & 2 & 1 & 2 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 1 & 0 & 3 & 2 & 4 \end{array}\right) \begin{array}{l} R_1 : \text{row } 1 \\ R_2 : \text{row } 2 \\ R_3 : \text{row } 3 \\ R_4 : \text{row } 4 \end{array}$$

Gaussian Elimination Algorithm

(a) Arrange so that the first number in the first row is not zero ($a_{11} \neq 0$)

✓ this is done

(b) use this number (a_{11}) as a "pivot" to eliminate coefficients of x_1 from the rest ~~a~~ of the rows/equations

($a_{21}, a_{31}, a_{41} = 0$)

Use [E3] $kL_1 + L_2 \to L_2$

Propose $-\frac{1}{4}L_1 + L_2 \to L_2$

$$\left(\begin{array}{cccc|c} 4 & 2 & 0 & 1 & 6 \\ 0 & 3/2 & 1 & 7/4 & -1/2 \\ 0 & 1 & 0 & 1 & 2 \\ 1 & 0 & 3 & 2 & 4 \end{array}\right)$$

$-\frac{1}{4}L_1 + L_4 \to L_4$

$$\left(\begin{array}{cccc|c} 4 & 2 & 0 & 1 & 6 \\ \boxed{0} & 3/2 & 1 & 7/4 & -1/2 \\ \boxed{0} & 1 & 0 & 1 & 2 \\ \boxed{0} & -1/2 & 3 & 7/4 & 10/4 \end{array}\right)$$

done!

(c) Examine each new row

(1) If a row is inconsistent (has the form $0 \cdot x_1 + 0 \cdot x_2 + \dots + 0x_n = b, b \neq 0$)

Stop $\Rightarrow$ no solution

(2) If the row is a multiple of another, then delete it (<u>one</u>, not both)

Repeat with "smaller" subsystem obtained by removing the first row

$$\left(\begin{array}{c|cccc|c} 4 & 2 & 0 & 1 & & 6 \\ \hline 0 & 3/2 & 1 & 7/4 & & -1/2 \\ 0 & 1 & 0 & 1 & & 2 \\ 0 & -1/2 & 3 & 7/4 & & 10/4 \end{array}\right)$$

new pivot point

subsystem

Propose $-\frac{2}{3}L_2 + L_3 \to L_3$

$$\left(\begin{array}{c|ccc|c} 4 & 2 & 0 & 1 & 6 \\ \hline 0 & 3/2 & 1 & 7/4 & -1/2 \\ 0 & 0 & -2/3 & -1/6 & 7/3 \\ 0 & -1/2 & 3 & 7/4 & 10/4 \end{array}\right)$$

$\frac{1}{3}L_2 + L_4 \to L_4$

$$\left(\begin{array}{c|ccc|c} 4 & 2 & 0 & 1 & 6 \\ \hline 0 & 3/2 & 1 & 7/4 & -1/2 \\ 0 & \boxed{0} & -2/3 & -1/6 & 7/3 \\ 0 & \boxed{0} & 10/3 & 7/3 & 7/3 \end{array}\right)$$

done!

new pivot point

Last step

$L_3 \cdot 5 + L_4 \to L_4$

Echelon form!

$$\left(\begin{array}{cccc|c} 4 & 2 & 0 & 1 & 6 \\ 0 & 3/2 & 1 & 7/4 & -1/2 \\ 0 & 0 & -2/3 & -1/6 & 7/3 \\ 0 & 0 & 0 & 18/12 & 14 \end{array}\right)$$

$$4x_1 + 2x_2 + x_4 = 6$$

$$3/2\, x_2 + x_3 + 7/4\, x_4 = -1/2$$

$$\vdots$$

$$\Rightarrow \boxed{x_4 = 14 \cdot \frac{12}{18}}$$