Lecture 5 Gussian Elimination

- Last week we discussed systems of linear equations and solution preserving elementary operations

Elementary operations

[E1] $L_i \leftrightarrow L_j$

[E2] $kL_i \rightarrow L_i$

[E3] $kL_i + L_j \rightarrow L_j$

Example
$$\begin{aligned} 4x_1 + 2x_2 + x_4 &= 6 \\ x_1 + 2x_2 + x_3 + 2x_4 &= 1 \\ x_2 + x_4 &= 2 \\ x_1 + 3x_3 + 2x_4 &= 4 \end{aligned}$$

We put into matrix form

$$\begin{array}{c} \left(\begin{array}{cccc|c} 4 & 2 & 0 & 1 & 6 \\ 1 & 2 & 1 & 2 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 1 & 0 & 3 & 2 & 4 \end{array}\right) \\ \begin{array}{ccccc} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \vec{a}_4 & \vec{b} \end{array} \end{array} \qquad \vec{a}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{m1} \end{bmatrix} \quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

a_{ij} coefficient of the j-th variable/unknown (x_j) in the i-th equation (L_i)

$i \in [0, m]$ m = # of equations

$j \in [0, n]$ n = # of unknown variables

Gaussian Elimination Algorithm

(a) Arrange so that the first number in the first row is not zero ($a_{11} \neq 0$)

✓ this is done

(b) Use this number (a_{11}) as a "pivot" to eliminate coefficients of x_1 from the rest of the rows/equations ($a_{21}, a_{31}, a_{41} = 0$)
Use [E3] $kL_1 + L_2 \to L_2$

Propose $-\frac{1}{4}L_1 + L_2 \to L_2$

$$\left(\begin{array}{cccc|c} 4 & 2 & 0 & 1 & 6 \\ 0 & 3/2 & 1 & 7/4 & -1/2 \\ 0 & 1 & 0 & 1 & 2 \\ 1 & 0 & 3 & 2 & 4 \end{array}\right)$$

Propose $-\frac{1}{4}L_1 + L_4 \to L_4$

$$\left(\begin{array}{cccc|c} 4 & 2 & 0 & 1 & 6 \\ 0 & 3/2 & 1 & 7/4 & -1/2 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & -1/2 & 3 & 7/4 & 10/4 \end{array}\right)$$

done!

c) Examine each new row

(1) If a row is inconsistent
(has the form $0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_4 = b$, $b \neq 0$)
stop $\Rightarrow$ no solution

(2) If the row is a multiple of another, then delete it
(one, not both)

Repeat with "smaller" subsystem obtained by removing the first row

new pivot point

$$\left(\begin{array}{cccc|c} 4 & 2 & 0 & 1 & 6 \\ \hline 0 & 3/2 & 1 & 7/4 & -1/2 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & -1/2 & 3 & 7/4 & 10/4 \end{array}\right)$$

Subsystem

propose

$$-\frac{2}{3} L_2 + L_3 \to L_3$$

$$\left(\begin{array}{cccc|c} 4 & 2 & 0 & 1 & 6 \\ 0 & 3/2 & 1 & 7/4 & -1/2 \\ 0 & 0 & -2/3 & -1/6 & 7/3 \\ 0 & -1/2 & 3 & 7/4 & 10/4 \end{array}\right)$$

$$\frac{1}{3} L_2 + L_4 \to L_4$$

$$\left(\begin{array}{cccc|c} 4 & 2 & 0 & 1 & 6 \\ 0 & 3/2 & 1 & 7/4 & -1/2 \\ 0 & \boxed{0} & -2/3 & -1/6 & 7/3 \\ 0 & \boxed{0} & 10/3 & 7/3 & 7/3 \end{array}\right)$$

done!

new subsystem

new pivot point

keep going!

$$L_3 \cdot \frac{10}{2} + L_4 \to L_4$$

$$\left(\begin{array}{cccc|c} 4 & 2 & 0 & 1 & 6 \\ 0 & 3/2 & 1 & 7/4 & -1/2 \\ 0 & 0 & -2/3 & -1/6 & 7/3 \\ 0 & 0 & 0 & 18/12 & 14 \end{array}\right)$$

Translate back to equations

$$4x_1 + 2x_2 + 0 \cdot x_3 + x_4 = 6$$
$$0 + 3/2\,x_2 + 1\,x_3 + 7/4\,x_4 = -1/2$$
$$0 + -2/3\,x_3 - 1/6\,x_4 = 7/3$$
$$18/12\,x_4 = 14$$

Solve for x_4

Solve for $x_3 = f(x_4)$

Solve for $x_2 = f(x_3, x_4)$

Solve for $x_1 = f(x_2, x_3, x_4)$

It is possible to solve the system of linear equasons by continuing to apply elementary operations.

Goal: make the pivot point equal to one, remove all possible off-diagonal coefficients (non-pivot points)

1st step I want to set all pivot points to one, Apply [E2]

Propose the following

$$\frac{1}{4} L_1 \to L_1$$

$$\frac{2}{3} L_2 \to L_2$$

$$-\frac{3}{2} L_3 \to L_3$$

$$\frac{2}{3} L_4 \to L_4$$

$$\left(\begin{array}{cccc|c} 1 & 1/2 & 0 & 1/4 & 6/4 \\ & 1 & 2/3 & 7/6 & -1/3 \\ & & 1 & 1/4 & -7/2 \\ & & & 1 & 28/3 \end{array}\right)$$

Now, we work backwards to introduce zeros on the off-diagonals

$$-\frac{1}{4} L_4 + L_3 \to L_3$$

$$\left(\begin{array}{cccc|c} 1 & 1/2 & 0 & 1/4 & 6/4 \\ & 1 & 2/3 & 7/6 & -1/3 \\ & & 1 & 0 & -7/3 - 7/2 \\ & & & 1 & 28/3 \end{array}\right)$$

$$-\tfrac{7}{6} L_4 + L_2 \to L_2$$

$$\left(\begin{array}{cccc|c} 1 & 1/2 & 0 & 1/4 & 6/4 \\ & 1 & 2/3 & 0 & -\frac{7 \cdot 14}{9} - 1/3 \\ & & 1 & 0 & -7/3 - 7/2 \\ & & & 1 & 28/3 \end{array}\right)$$

$$-\tfrac{1}{4} L_4 + L_1 \to L_1$$

done!

new subsystem →

$$\left(\begin{array}{ccc|c|c} 1 & 1/2 & 0 & 0 & -\frac{1}{4} \cdot \frac{28}{3} + 6/4 \\ & 1 & 2/3 & 0 & -7 \cdot 14/9 - 1/3 \\ \hline & & 1 & 0 & -7/3 - 7/2 \\ & & & 1 & 28/3 \end{array}\right)$$

$$-\tfrac{2}{3} L_3 + L_2 \to L_2$$

$$\left(\begin{array}{cccc|c} 1 & 1/2 & 0 & 0 & 1/4 \cdot \frac{28}{3} + \frac{6}{4} \\ & 1 & 0 & 0 & -84/9 + 6/3 \\ & & 1 & 0 & -7/3 - 7/2 \\ & & & 1 & 28/3 \end{array}\right)$$

$-\frac{1}{2}L_2 + L_1 \to L_1$

$$\left[\begin{array}{cccc|l} 1 & 0 & 0 & 0 & -\frac{1}{2}\left(-\frac{84}{9}+6/3\right)-\frac{28}{12}+6/4 \\ & 1 & 0 & 0 & -84/9+6/3 \\ & & 1 & 0 & -7/3-7/2 \\ & & & 1 & 28/3 \end{array}\right]$$

<u>row canonical form</u>

Linear independence $\Leftrightarrow$ triangular form

$\Leftrightarrow$ unique solution

Take L_1 for example, does there exist k_1, k_2, k_3 s.t.

$$k_1 L_2 + k_2 L_3 + k_3 L_4 = L_1$$

If yes; linearly dependent

If no: linearly independent