5  Week 3 Monday

5.1 Transforms
Transforms provide a change to a more suitable basis. Integral transform maps equation from its original domain into another domain. This can make a problem easier to solve. An integral transform is given by an expression of the form

\[ F(x) = I[f](x) = \int_a^b k(x, x') f(x') dx' \]

where \( k(x, x') \) is known as the kernel.

5.2 Heat equation on an infinite domain
\[ \phi_{xx} + \lambda \phi = 0 \]
Recall it was the boundary conditions on a finite domain that led to discrete eigenvalues \( \lambda = (n\pi/L)^2 \) and a Fourier series expansion.

Consider an infinite length domain, then we need BCs as \( x \to \infty \). We would expect \( \phi(x) \to 0 \) as \( x \to \infty \) but for now we relax the condition to \( |\phi(x)| < \infty \) to avoid arriving only at a trivial solution, however the former still holds.

We still have the condition \( \lambda > 0 \). However, now we have a continuous spectrum of eigenvalues. Any non-negative values of \( \lambda \) are allowed. Because we no longer have integer only values, we take an integral (analogous to sum) over all the possible values of \( \lambda \).

Recall from the heat equation we had a solution of the form

\[ u = e^{-i\omega x} e^{-\lambda^2 t} \]
where \( \lambda = \omega^2 \). The function \( e^{-i\omega x} \) with \( \omega \in (-\infty, \infty) \) is the equivalent (compact form) of

\[ u = c_1 \cos(\sqrt{\lambda} x) + c_2 \sin(\sqrt{\lambda} x) \]

In either form "summing" (i.e. integrating) over the continuous spectrum of eigenvalues gives

\[ u(x, t) = \int_{-\infty}^{\infty} [A(\omega) \cos(\omega x) e^{-\omega^2 t} + B(\omega) \sin(\omega x) e^{-\omega^2 t}] d\omega \]
and equivalently

\[ u(x, t) = \int_{-\infty}^{\infty} c(\omega) e^{-i\omega x} e^{-\omega^2 t} \]

Where the Fourier series expansion allows one to express a function as a sum of functions each pertaining to discrete eigenvalues. A Fourier transform will allow us to represent a function as an integral over a continuous spectrum of eigenvalues. It’s a change of basis to the frequency domain.

5.3 Fourier transform
Note that the following definition is used in the book

\[ F(w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\bar{x}) e^{iwx} d\bar{x} \]
\[
\frac{f(x^+) + f(x^-)}{2} = \int_{-\infty}^{\infty} F(\omega)e^{-i\omega x} d\omega
\]

In this class for continuous differentiable \( f(x) \) we define the Fourier Transform

\[
F(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x} dx
\]

and inverse transform by

\[
f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)e^{i\omega x} d\omega.
\]

We denote the operator by \( \mathcal{F}[f(x)](\omega) = F(\omega) \).

Note: the convolution theorem tells us that if \( h(x) = (f \ast g)(x) = \int_{-\infty}^{\infty} f(y)g(x-y)dy \)
where \( \ast \) denotes the convolution operation, then: \( H(k) = F(k) \cdot G(k) \)
and the Fourier transform of \( h(x) \), \( g(x) \), and \( y(x) \), respectively.

We know look a the Fourier transform of some common functions. Consider the indicator function

\[
I_a(x) = \begin{cases} 
1 & |x| < a \\
0 & |x| > a 
\end{cases}
\]

We take the Fourier transform

\[
F(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x} dx
\]

\[
= \int_{-a}^{a} e^{-i\omega x} dx = \frac{e^{-i\omega a} - e^{i\omega a}}{-i\omega}
\]

\[
= 2\sin(a\omega) \frac{\omega}{\omega}
\]

We can apply the inver Fourier transform to get back the indicator function:

\[
I_a(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\sin(a\omega) e^{i\omega x} d\omega
\]

Let’s look at the Dirac delta function

\[
\mathcal{F}[\delta(x-x_0)] = \int_{-\infty}^{\infty} \delta(x-x_0)e^{i\omega x} dx = e^{-i\omega x_0}
\]

\[
\mathcal{F}^{-1}[\delta(\omega-\omega_0)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega-\omega_0)e^{i\omega x} d\omega = \frac{1}{2\pi} e^{i\omega x_0}
\]

It follows that

\[
\delta(x-x_0) = \mathcal{F}^{-1}[\mathcal{F}[\delta(x-x_0)]] = \mathcal{F}^{-1}[e^{-i\omega x_0}]
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x_0} e^{i\omega x} d\omega
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega(x-x_0)} d\omega
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega x_0} e^{i\omega x} d\omega.
\]
which is consistent with what we found above.

Let’s consider the Fourier transform of the Heaviside step function

\[ H(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \]

\[ \mathcal{F}[H] = \hat{H}(\omega) = \int_{-\infty}^{\infty} H(t)e^{-i\omega t} dt = \int_{0}^{\infty} e^{-i\omega t} dt \]

Let \( G(\omega) = \hat{H}(\omega) \). We take a different approach

\[ G_a(\omega) = \int_{-\infty}^{\infty} e^{-at} H(t)e^{-i\omega t} dt \]

\[ G_a(\omega) = \int_{0}^{\infty} e^{-(a+i\omega)t} dt \]

\[ = \left. e^{-(a+i\omega)t} \right|_{t=0}^{t=\infty} \]

\[ = \frac{1}{a + i\omega} \]

\[ \implies G_{a=0}(\omega) = \frac{1}{i\omega} \]

\[ \mathcal{F}[H(t)e^{-at}] = \frac{1}{a + i\omega} \]

We now apply the Fourier transform to solve an ODE

\[ g''(x) - g(x) = \delta(x) \]

Let \( G(k) = \mathcal{F}[g](k) \), then

\[ g(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(k)e^{ikx} dk \]

\[ g' = \frac{d}{dx} \int_{-\infty}^{\infty} G(k)e^{ikx} dk \]

\[ = \int_{-\infty}^{\infty} G(k) \left[ \frac{d}{dx} e^{ikx} \right] dk \]

\[ = \int_{-\infty}^{\infty} G(k) (ik)e^{ikx} dk \]

thus,

\[ \mathcal{F}[g'] = (ik)G(k) \]

and similarly it can be shown

\[ \mathcal{F}[g''] = (ik)^2 G(k) \]

So if we apply the Fourier transform on the ODE we get

\[ (ik)^k G(k) - G(k) = 1 \]
solving for $G(k)$ we get

$$G(k) = -\frac{1}{1 + k^2}.$$ 

Now we can find $g(x)$ through the inverse Fourier transform

$$g(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ikx}}{1 + k^2} dk.$$ 

Note we can rewrite $G(k)$ as

$$G(k) = \frac{-1/2}{1 + ik} + \frac{-1/2}{1 - ik}$$

and from previous results we know

$$g(x) = -\frac{1}{2} e^{-x} H(x) - \frac{1}{2} e^{x} H(-x)$$

(135)

$$= -\frac{1}{2} e^{\vert x \vert}$$

(136)

Consider the ODE

$$y''(x) = y(x) = f(x)$$

We can solve this equation using our previous results. The response to the Dirac delta function is known as Green’s function.

**Green’s function** A Green’s function is the impulse response of an inhomogeneous linear differential equation defined on a domain, with specified initial conditions or boundary conditions. Since the impulse function contains all frequencies, the impulse response defines the response of a linear time-invariant system for all frequencies. While it is impossible to apply a dirac delta function to any real system, it is a useful idealization and could even be approximated. In Fourier analysis theory, such an impulse comprises equal portions of all possible excitation frequencies, which makes it a convenient test probe. Any system in a large class known as linear, time-invariant (LTI) is completely characterized by its impulse response. That is, for any input, the output can be calculated in terms of the input and the impulse response.

If you know the Green’s function you can then find the solution to system with forcing function $f(x)$

$$u(x) = \int G(x, s)f(s)ds$$

assuming the system can be defined by a linear operator

$$L[G(x, x_s)] = \delta(x - x_s).$$

We solved the response to a delta function now we can find the response to

$$y''(x) - y(x) = f(x).$$

We take the Fourier transform

$$(ik)^2 Y(k) - Y(k) = F(k)$$

and solve for $Y(k)$

$$Y(k) = \frac{-1}{1 + k^2} F(k)$$
then we know
\[ y(x) = \mathcal{F}^{-1}[Y](x) = -\mathcal{F}^{-1}\left[ \frac{1}{1 + k^2} \right] \ast f(x) \]
\[ = -\frac{1}{2} e^{|x|} \ast f(x) \]
\[ = \int_{-\infty}^{\infty} f(\xi) \left[ -\frac{1}{2} e^{-|x-\xi|} \right] d\xi \]
\[ = -\frac{1}{2} \int_{-\infty}^{\infty} f(\xi)e^{x-\xi}d\xi \]

5.4 Bonus Example

Consider the following inhomogeneous wave equation
\[ v_{tt} - c^2 v_{xx} = k\delta(x - L/2)\delta(t - T) \]
\[ v_x(0, t) = v_x(L, t) = 0 \]
\[ v(x, 0) = U(L/2 - x) \]
\[ v_t(x, 0) = 0 \]

Here, \( U \) is the Heaviside step function.

First solve the homogeneous PDE to find a complete set of eigenfunctions that satisfy the boundary conditions.
\[ v_{tt} - c^2 v_{xx} = 0 \]

Let \( v(x, t) = X(x)T(t) \), the from separation of variables we find
\[ \frac{1}{c^2} \frac{T_{tt}}{T} = \frac{X_{xx}}{X} = -\lambda^2 \]
The ODE for \( X \) is an S-L eigenvalue problem. We know the eigenfunctions for \( X \) take the form \( X = A\cos(\lambda x) + B\sin(\lambda x) \) and after applying the BC, we find that \( X_n = \cos \left( \frac{n\pi x}{L} \right) \) for \( n = 0, 1, 2, \ldots \).

Each eigenfunction satisfies the boundary condition so \( V \), which is a sum of \( X_n \)'s, will automatically satisfy the BC's.
\[ V = \sum_{n=0}^{\infty} \alpha_n(t) \cos \left( \frac{n\pi x}{L} \right) \]

First, we match the initial conditions
\[ V(x, 0) = \sum_{n=0}^{\infty} \alpha_n(0) \cos(n\pi x/L) = U(L/2 - x) \]
\[ V_t(x, 0) = \sum_{n=0}^{\infty} \alpha'_n(0) \cos(n\pi x/L) = 0 \implies \alpha'_n(0) = 0 \]

Multiply both sides of the first equation by \( \cos(m\pi x/L) \) and integrate from 0 to \( L \) to get
\[ \sum_{n=0}^{\infty} \alpha_n(0) \int_{0}^{L} \cos(n\pi x/L) \cos(m\pi x/L) dx = \int_{0}^{L} \cos(m\pi x/L)U(L/2 - x) dx \]
\[ \alpha_m(0) \frac{L}{2} = \int_{0}^{L/2} \cos(m\pi x/L) dx \]
So we get the boundary values

\[
\alpha_n(0) = \frac{2}{n\pi} \sin\left(\frac{n\pi}{2}\right) \quad \text{for } n = 1, 2, 3, \ldots \tag{150}
\]

\[
\alpha_0(0) = 1/2. \tag{151}
\]

Therefore

\[
V(x, 0) = 1/2 + \frac{2}{\pi} \sum_{n=1}^{\infty} \left(\sin\left(\frac{n\pi}{2}\right)/n\right) \cos\left(n\pi x/L\right)
\]

Now we deal with the inhomogeneous part of the problem. Expand \(\delta(x - L/2)\) in terms of the eigenfunctions

\[
\beta_0 + \sum_{n=1}^{\infty} \beta_n \cos(n\pi x/L) = \delta(x - L/2)
\]

Then multiplying both side by \(\cos(m\pi x/L)\) and integrating over \(0\) to \(L\), we find the coefficients \(\beta_n\)

\[
\beta_n = 2 \cos\left(m\pi/2\right)/L \quad \text{for } n = 1, 2, 3, \ldots \tag{152}
\]

\[
\beta_0 = 1/2. \tag{153}
\]

We plug our series expression for \(V(x, t)\) along with our series expansion of \(\delta(x - L/2)\) back into our PDE \(v_{tt} - c^2v_{xx} = k\delta(x - L/2)\delta(t - T)\). Grouping like terms we get

\[
\left(\alpha''_0(t) - \kappa\delta(t - T)/L\right) + \sum_{n=1}^{\infty} \left[\alpha''_n(t) + \frac{c^2\pi^2 n^2}{L^2} \alpha_n(t) - \frac{2\cos(n\pi/2)}{L} \kappa\delta(t - T)\right] \cos(n\pi x/L) = 0
\]

This gives an ODE for each \(\alpha_n\) and remember we already found their corresponding IC to solve the ODE.

\[
\alpha''_0(t) = \kappa\delta(t - T)/L \tag{154}
\]

\[
\alpha''_n(t) + \frac{c^2\pi^2 n^2}{L^2} \alpha_n(t) = \frac{2\cos(n\pi/2)}{L} \kappa\delta(t - T) \tag{155}
\]

For the first equation we can integrate twice and applying the boundary conditions \(\alpha_0(0) = 1/2\) and \(\alpha'_0(0) = 0\) we get

\[
\alpha_0(t) = \frac{1}{2} + \frac{\kappa(t - T)U(t - T)}{L}
\]

If you solve the second equation you will find

\[
\alpha_n(t) = \frac{2\sin(n\pi/L)}{n\pi} \cos(cn t/L) + \frac{2\cos(n\pi/2)}{cn\pi} \sin\left(\frac{n\pi n}{L}\right) (t - T)U(t - T)
\]

which can be found using the Greens function approach.