Lecture 6

$k \le m$

~~Gauss~~

$\boxed{\begin{matrix} L_1, \dots, L_m \\ x_1, \dots, x_n \end{matrix}} \xrightarrow{\text{Gaussian elimination}} \boxed{\begin{matrix} L_1, \dots, L_k \\ x_1, \dots, x_n \end{matrix}} \longrightarrow$

Original System — equivalent System

$\searrow$ no solution

inconsistent equation

$\longrightarrow \boxed{\begin{matrix} L_1, \dots, L_k \\ x_1, \dots, x_n \end{matrix}}$ $\nearrow$ $k = n$ one solution

$\searrow$ $k < n$ infinite solutions

Example

$$\begin{aligned} x_2 + x_3 &= 6 \\ x_1 + 4x_2 + 3x_3 &= 17 \\ x_1 + 2x_2 + x_3 &= 5 \\ x_1 + 3x_2 + 2x_3 &= 11 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 0 & 1 & 1 & 6 \\ 1 & 4 & 3 & 17 \\ 1 & 2 & 1 & 5 \\ 1 & 3 & 2 & 11 \end{array}\right)$$

a) does $a_{11} = 0$?

$L_1 \leftrightarrow L_2$

$$\left(\begin{array}{ccc|c} 1 & 4 & 3 & 17 \\ 0 & 1 & 1 & 6 \\ 1 & 2 & 1 & 5 \\ 1 & 3 & 2 & 11 \end{array}\right)$$

$$-L_1 + L_3 \rightarrow L_3$$

$$\left(\begin{array}{ccc|c} 1 & 4 & 3 & 17 \\ 0 & 1 & 1 & 6 \\ \cancel{0} & \cancel{-2} & \cancel{-2} & \cancel{-12} \\ 1 & 3 & 2 & 11 \end{array}\right)$$

note $L_3 = -2L_1$ delete one

$$-L_1 + L_3 \rightarrow L_3$$

$$\left(\begin{array}{ccc|c} 1 & 4 & 3 & 17 \\ 0 & 1 & 1 & 6 \\ 0 & -1 & -1 & -6 \end{array}\right)$$

$L_3 = -L_2$ delete one

$$\text{pivot} \rightarrow \left(\begin{array}{ccc|c} 1 & 4 & 3 & 17 \\ 0 & 1 & 1 & 6 \end{array}\right) \quad \text{echelon form}$$

pivot

free variable

$$x_1 + 4x_2 + 3x_3 = 17$$

$$x_2 + x_3 = 6$$

Free to choose x_3, then solve for x_1, x_2

I designed this system to have 2 pivots and 1 free variable the following way

I started with two linearly independent equations

L_1 $\qquad x_2 + x_3 = 6$

L_2 $\qquad x_1 + 4x_2 + 3x_3 = 17$

I then created one more equation using the two above

L_3
$(L_2 - 2L_1)$ $\qquad x_1 + 2x_2 + x_3 = 5$ (linearly dependent on L_1, L_2)

Then I create a fourth equation by adding $L_1 + L_3$

$x_1 + 3x_2 + 2x_3 = 11$ (linearly dependent on L_1, L_3)

This ensures
2 pivot points and 1 free variable

free variables = # unknown − # ~~pid~~ pivots

$\qquad n=3 \qquad\qquad 2$

$= 3 - 1 = 1$

Let's return to the matrix representation

From Tuesday's example, the augmented matrix for our system

$$\left(\begin{array}{cccc|c} 4 & 2 & 0 & 1 & 6 \\ 1 & 2 & 1 & 2 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 1 & 0 & 3 & 2 & 4 \end{array}\right)$$

$$\underbrace{}_{\vec{a}_1} \quad \underbrace{}_{\vec{a}_2} \quad \underbrace{}_{\vec{a}_3} \quad \underbrace{}_{\vec{a}_4} \quad \underbrace{}_{\vec{b}}$$

Note that we can write the system of linear equations as a linear combination of vectors

$$\vec{a}_1 x_1 + \vec{a}_2 x_2 + \vec{a}_3 x_3 + \vec{a}_4 x_4 = \vec{b}$$

$$\begin{bmatrix} 4 \\ 1 \\ 0 \\ 1 \end{bmatrix} x_1 + \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \end{bmatrix} x_3 + \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix} x_4 = \begin{bmatrix} 6 \\ 1 \\ 2 \\ 4 \end{bmatrix}$$

we solved x_1, x_2, x_3, x_4

Plug in $x_4 = 28/3$

$x_3 = -7/3 - 7/2$

$x_2 = -84/9 + 6/3$

$x_1 = -\frac{1}{2}\left(\frac{-84}{9} + 6/3\right)$

$-28/12 + 6/4$

We can concatenate vectors to construct a matrix

$$A = [\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4]$$

But this is just the matrix representation for Gaussian elimination

Matrix is a rectangular array of elements. A matrix having m rows and n columns is referred to as an $m \times n$ matrix

If $m = n$, the matrix is called a square matrix

Then we can write the system in the following compact form

$$A\vec{x} = \vec{b} \quad \text{where } \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \vec{b} = \begin{bmatrix} 6 \\ 1 \\ 2 \\ 4 \end{bmatrix}$$

$A\vec{x}$ denotes the product of matrix A and vector $\vec{x}$

If A is an $m \times n$ matrix then this operation is only defined if $\vec{x}$ is a vector of dimension $n \times 1$

The i-th component of $A\vec{x}$ is denoted by $[A\vec{x}]_i$

$$A\vec{x} = \begin{bmatrix} \\ \square \\ \\ \end{bmatrix} \leftarrow \text{i-th component}$$

$$[A\vec{x}]_i = \sum_{j=1}^{n} a_{ij} x_j \qquad a_{ij} = [A]_{ij}$$

for $i = 1, 2, \dots, m$

Example
$$3x_1 + 2x_2 = 1$$
$$x_1 - x_2 = 5$$

Corresponding matrix A is

$$A = \begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix}$$

$n = 2$ unknowns

$m = 2$ equations

A is an $m \times n$ matrix

2×2 matrix

1st component: $[A\vec{x}]_1 = \sum_{j=1}^{n} a_{1j} x_j = a_{11} x_1 + a_{12} x_2$
$$= 3x_1 + 2x_2$$

2nd component: $[A\vec{x}]_2 = \sum_{j=1}^{n} a_{2j} x_j = a_{21} x_1 + a_{22} x_2$
$$= x_1 + -x_2$$

resulting vector $\vec{v} = A\vec{x} = \begin{bmatrix} [A\vec{x}]_1 \\ [A\vec{x}]_2 \end{bmatrix} = \begin{bmatrix} 3x_1 + 2x_2 \\ x_1 - x_2 \end{bmatrix}$

$$A\vec{x} = \vec{b}$$

$$\Rightarrow \begin{bmatrix} 3x_1 + 2x_2 \\ x_1 - x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

~~The linear~~ we get the system of linear equations in vector format

Examples of products

$$\begin{bmatrix} 1 & 2 \\ 4 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1x_1 + 2x_2 \\ 4x_1 + 10x_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2\cdot 1 + 1\cdot 1 \\ 6 + 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 6\cdot 3 + 2\cdot 2 \\ 0\cdot 3 + 1\cdot 2 \end{bmatrix} = \begin{bmatrix} 22 \\ 2 \end{bmatrix}$$

We can use non-square matrix

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1\cdot 1 + 2\cdot 2 \\ 3\cdot 1 + 2\cdot 6 \\ 4\cdot 1 + 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 15 \\ 4 \end{bmatrix}$$

$$(3\times 2) \quad (2\times 1) = (3\times 1)$$

$$\begin{bmatrix} 1 & 2 & 8 \\ 3 & 6 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \times$$

$$(2\times 3) \quad (2\times 1)$$

not defined

Properties of $A\vec{x}$

$$A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v}$$
$$A(c\vec{u}) = cA\vec{u}$$
$$A\vec{0} = \vec{0}$$

Consider the case of a square matrix

Recall Tuesday's example

$$\left(\begin{array}{c|c} A & \vec{b} \end{array}\right) \xrightarrow[\text{operations}]{\text{elementary}} \left(\begin{array}{c|c} \underbrace{\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}}_{\text{This is known as the identity matrix}} & \begin{matrix} \tilde{b}_1 \\ \tilde{b}_2 \\ \tilde{b}_3 \\ \tilde{b}_4 \end{matrix} \end{array}\right)$$

Some properties of identity matrix

$$AI = IA = A$$
$$Iv = v$$

It's analogous to multiplying by 1!

We can think of the set of elementary operations as transformations

transforming what?

$A\vec{x} = \vec{b} \Rightarrow \vec{a}_1 x_1 + \vec{a}_2 x_2 + \vec{a}_3 x_3 + \vec{a}_4 x_4 = \vec{b}$

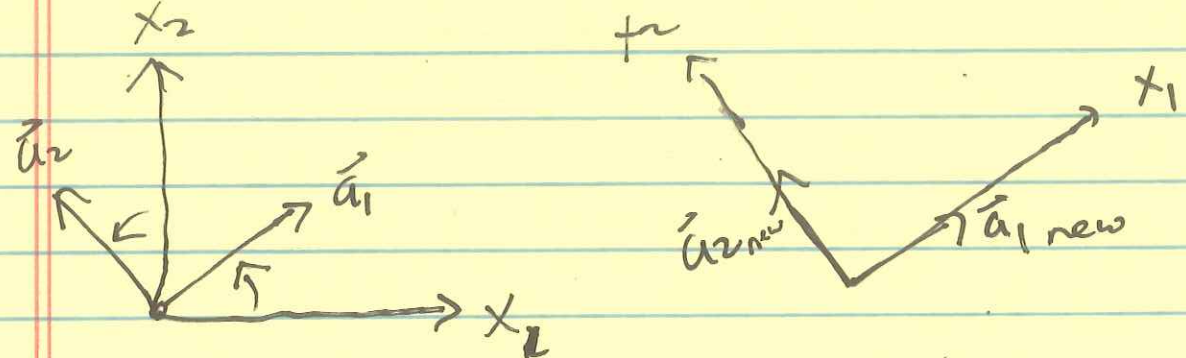
New system

$A_{new} \vec{x} = \vec{b}_{new}$

Identity matrix

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} x_4 = \vec{b}_{new}$$

It's like transforming your coordinate system in n-dimensional space



If the matrix A is an $n \times n$ square matrix and has n linearly independent rows (or columns) then there exist a matrix B. s.t.

$$BA = I$$

B is then called the inverse of A and denoted by A^{-1}

A^{-1} contains all the information to map $A \rightarrow I$ (all the elementary operations)