Lecture 6

$$\boxed{\begin{matrix} L_1, \dots, L_m \\ x_1, \dots, x_n \end{matrix}} \xrightarrow{\text{Gaussian Elimination}} \boxed{\begin{matrix} L_1, \dots, L_k \\ x_1, \dots, x_n \end{matrix}} \quad k \leq m$$

$\searrow$ no solution
inconsistent

$$\Rightarrow \boxed{\begin{matrix} L_1, \dots, L_k \\ x_1, \dots, x_n \end{matrix}}$$

$\rightarrow k = n$ one solution n pivot variables

$\searrow k < n$ infinite solution

free variables ≥ 1

Try a different representation

Example:

$$x_2 = 6$$
$$x_1 + 2x_2 = 1$$

$$\left(\begin{array}{cc|c} 0 & 1 & 6 \\ 1 & 2 & 1 \end{array}\right) \Rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 2 \end{bmatrix} x_2 = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$$

We can write the system of linear equations as a linear combination of vectors

$$\vec{a}_1 x_1 + \vec{a}_2 x_2 + \dots + \vec{a}_n x_n = \vec{b}$$

Linear Spans

Suppose $\vec{a}_1, .., \vec{a}_n$ are any vectors in a vector space $\mathbb{R}^m$ ← set of all m-dimensional vectors

The collection of all linear combinations

$$x_1\vec{a}_1 + x_2\vec{a}_2 + .. + x_n\vec{a}_n \quad \text{where}$$

x_i are scalars $(x_i \in \mathbb{R})$

denoted by

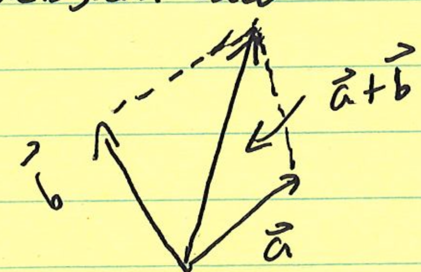
$$\text{span}(\vec{a}_1, \vec{a}_2, ..., \vec{a}_n) \text{ or } \text{span}(\vec{a}_i)$$

is called the linear span of $\vec{a}_1, .., \vec{a}_n$

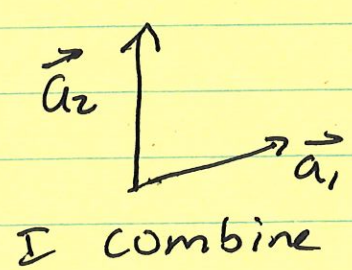
Zero vector belongs to the $\text{span}(\vec{a}_i)$, since $0 = 0 \cdot \vec{a}_1 + 0 \cdot \vec{a}_2 + ... + 0 \cdot \vec{a}_n$

Return to 2D example

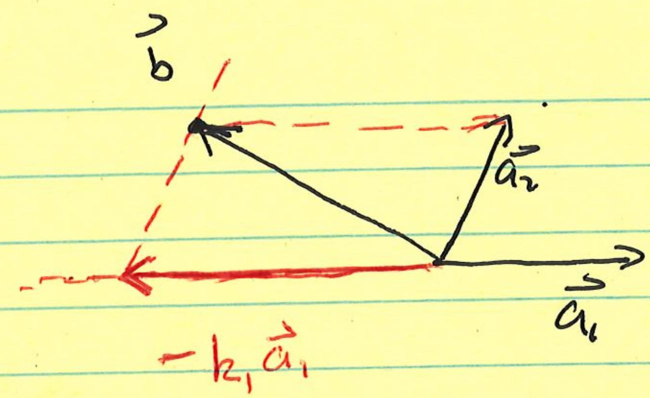
from parallelogram law



If I have two vectors $\vec{a}_1, \vec{a}_2 \in \mathbb{R}^2$



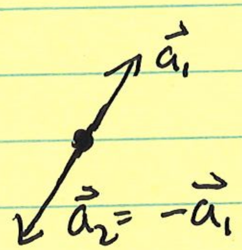
how can I combine them to get to my desired vector $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

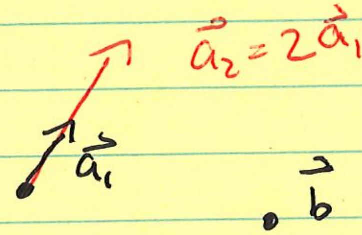


$\vec{b} = -k_1\vec{a}_1 + \vec{a}_2$

flip $\vec{a}_1$
stretch $\vec{a}_1$
Leave $\vec{a}_2$ as is

Suppose our two vectors $\vec{a}_1$ and $\vec{a}_2$ where in line with each other





Question: If I have a set of vectors $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n$ what are the possible $\vec{b}$'s I can get

$\Rightarrow \vec{b} \in \text{span}(\vec{a}_i)$
"in"

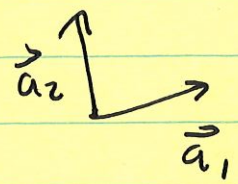
non overlapping in direction — Summary: We found that if we have two vectors $\vec{a}_1, \vec{a}_2 \in \mathbb{R}^2$ ~~then~~ such that

$\text{span}\{\vec{a}_1, \vec{a}_2\} = \mathbb{R}^2$

"there exists" $\exists\; x_1, x_2$ s.t. $x_1\vec{a}_1 + x_2\vec{a}_2 = \vec{b} \;\; \forall \;\vec{b} \in \mathbb{R}^2$ "for all"

"such that"

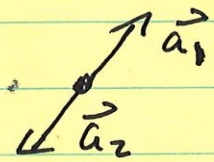
In the first case



$\vec{a}_1$ and $\vec{a}_2$ can be linearly combined to produce any other vector in 2D

We say $\vec{a}_1$ and $\vec{a}_2$ generates (or spans) the entire 2-dimensional space $\mathbb{R}^2$ (all 2×1 vectors)

In the second case



$\vec{a}_1$ and $\vec{a}_2$ generate or span a subspace of $\mathbb{R}^{2\times 1}$ ($\mathbb{R}^2$) but not the complete vector space V

For example:

Consider the subspace generated by

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \text{span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = W$$

Note that

$$W = \text{Span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}\right)$$

also,

$$W = \text{Span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \end{bmatrix}\right)$$

The last two examples are linearly dependent sets

Definition: We say that vectors $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ in $\mathbb{R}^m$ are <u>linearly dependent</u> if there exist scalars $k_1, \ldots, k_n$ not all zero, such that

$$k_1\vec{a}_1 + k_2\vec{a}_2 + \ldots + k_n\vec{a}_n = 0$$

Otherwise, we say that they are linearly independent

$$\vec{a}_1 = \begin{bmatrix} a \\ b \end{bmatrix} = [a \; b]^T$$

$$\vec{a}_2 = \frac{-k_1\vec{a}_1 - k_3\vec{a}_3 \ldots - k_n\vec{a}_n}{k_2}$$

If the question is to find a <u>unique</u> linear combination then check for two things:

- check for linear independence (of the set)
- check that the vector is in the span of the set

Consider the system

$$x_1 - 2x_2 = b_1$$
$$2x_1 - 4x_2 = b_2$$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} x_1 + \begin{bmatrix} -2 \\ -4 \end{bmatrix} x_2 = \underbrace{\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}}_{\vec{b}}$$

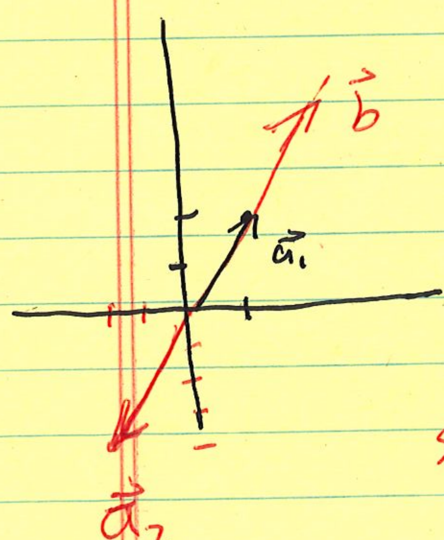
$$\left(\begin{array}{cc|c} 1 & -2 & b_1 \\ 2 & -4 & b_2 \end{array}\right)$$

$$-2L_1 + L_2 \rightarrow L_2$$

free variable

pivot point

$$\left(\begin{array}{cc|c} 1 & -2 & b_1 \\ 0 & 0 & \underbrace{-2b_1 + b_2}_{\vec{b}} \end{array}\right)$$



If $-2b_1 + b_2 = 0$, then the system is consistent

$$\Rightarrow b_2 = 2b_1$$

$$\text{span}\{\vec{a}_1, \vec{a}_2\} = \text{span}\{\vec{a}_1\} = W$$

$\vec{b} \in W$

$$W = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} : b = 2a \ \forall a \in \mathbb{R} \right\}$$

Infinite solutions (1 free variable)

$\Rightarrow$ Infinite ways to generate the vector $\vec{b}$ for $b_2 = 2b_1$ (otherwise no solution)