Lecture 7

Summary: thus far we have discussed methods of analyzing and solving a system of linear equations of the form

$$L_1: \; a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
$$L_2: \; a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$
$$\vdots$$
$$L_m: \; a_{m1}x_1 + a_{m2}x_2 + \dots + a_{2n}x_n = b_n$$

We put this in the augmented matrix form

$$m \left(\begin{array}{cccc|c} \multicolumn{4}{c|}{\overbrace{\hphantom{a_{11} \; a_{12} \; \dots \; a_{1n}}}^{A}} & \vec{b} \\ a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & & \ddots & \vdots & \vdots \\ a_{m1} & \dots & \dots & a_{mn} & b_m \end{array} \right)$$
$$n$$

which allowed us to more readily apply elementary row operations

$$L_i \leftrightarrow L_j \;, \; kL_i \to L_i \;, \; kL_i + L_j \to L_j$$

to reduce the system to <u>echelon form</u>

Echelon form reveals many properties of the system including:
- existence and uniqueness of solutions
- linear dependence/independence of $L_1, \dots, L_m$
- # o' pivot variables and free variables

Continuing to apply elementary operations until the pivot points (first non-zero element in each row) is the only non-zero term in the column gives us row canonical form (reduced row echelon form rref).

Last lecture we considered the special case where in echelon form the number of equations and unknowns are equal

Looks like..

$$\left(\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & \tilde{b}_1 \\ 0 & 1 & & \vdots & \tilde{b}_2 \\ \vdots & & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & \tilde{b}_n \end{array}\right)$$

$\underbrace{\qquad\qquad}_{I}$

n equations

n unknowns

in row canonical form

Then, it is clear that a unique solution exists

The matrix I is the identity matrix.

The identity matrix is such that

$$a_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

So there are only 1's on the diagonal and the rest are 0's

Last lecture we introduced the concept of the inverse

A square matrix A is said to be invertible or non-singular if there exists a matrix B such that

$$AB = BA = I \qquad B, I, A \in \mathbb{R}^{n \times n}$$

Square matrices of dimension n

where I is the identity matrix. Such a matrix B is unique. B is called the inverse of A denoted by A^{-1}

Side on matrix algebra

The product AB is defined as follows, if $a_{ij} = [A]_{ij}$ and $b_{ij} = [B]_{ij}$ then

$$[C]_{ij} = [AB]_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

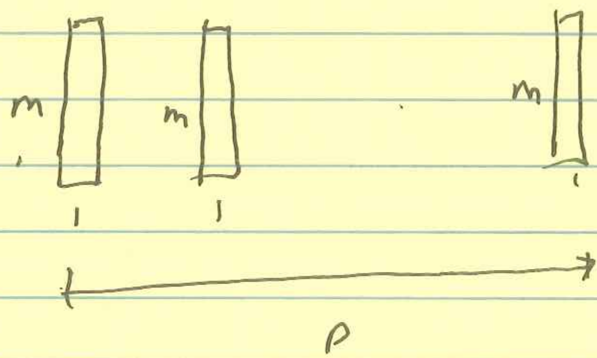
$$\underset{(m \times n)}{A} \; \underset{(n \times p)}{B} = \underset{(m \times p)}{C}$$

have to be the same

Suppose $B = [\vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_4, \dots, \vec{b}_p]$ $\vec{b}_i \in \mathbb{R}^{m \times 1}$

$$AB = [A\vec{b}_1, A\vec{b}_2, \dots, A\vec{b}_p]$$

$(m \times n)(n \times p) =$



$(m \times p)$

Back to our discussion

What is the role of the inverse?

The inverse contains all the elementary row operations required to get to row canonical form

$$\underbrace{A^{-1}}_{\text{inverse}} A = \underbrace{I}_{\text{identity}}$$

Why is this useful? Well if we know A^{-1} we can solve the system $A\vec{x} = \vec{b}$

$$A^{-1}(A\vec{x} = \vec{b})$$

$$A^{-1}A\vec{x} = A^{-1}\vec{b}$$

by definition $\longrightarrow$ $I\vec{x} = A^{-1}\vec{b}$

$$\boxed{\vec{x} = A^{-1}\vec{b}}$$

Goals: Calculate the inverse

One method involves the exact method applied to reach row canonical form

We start with the following augmented matrix

$$\left(\begin{array}{cccc|cccc} a_{11} & a_{12} & \dots & a_{1n} & 1 & 0 & \dots & 0 \\ a_{21} & & & \vdots & 0 & 1 & \ddots & \vdots \\ \vdots & & & \vdots & \vdots & & \ddots & \vdots \\ & & & & 0 & & & \\ a_n & & & a_{mn} & 0 & \dots & & 0 \; 1 \end{array}\right)$$

$\underbrace{\phantom{a_{11} \dots a_{mn}}}_{A}$ note: requires a square matrix

Example

$$\left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right]$$

$$-2L_1 + L_2 \to L_2$$

$$\left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & -7 & -2 & 1 \end{array}\right]$$

$$-\frac{1}{7}L_2 \to L_2$$

$$\left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & 1 & 2/7 & -1/7 \end{array}\right]$$

$-4L_2 + L_1 \to L_1$

$$\left[\begin{array}{cc|cc} 1 & 0 & -\frac{8}{7}+1 & 4/7 \\ 0 & 1 & 2/7 & -1/7 \end{array}\right]$$

$$\underbrace{\qquad\qquad}_{A^{-1}}$$

Let's try it out!

$$A^{-1}A = \begin{bmatrix} -1/7 & 4/7 \\ 2/7 & -1/7 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 1 \end{bmatrix}$$

$$A^{-1}\begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad A^{-1}\begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$= \left[\begin{array}{c|c} -\frac{1}{7} + 8/7 & -\frac{4}{7} + \frac{4}{7} \\ 2/7 - \frac{2}{7} & \frac{8}{7} - \frac{1}{7} \end{array}\right]$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

yay!

Lets solve the system

$$A\vec{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad \vec{x} = A^{-1} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1/7 & 4/7 \\ 2/7 & -1/7 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{2}{7} + \frac{4}{7} \\ \frac{4}{7} - \frac{1}{7} \end{bmatrix} = \begin{bmatrix} 2/7 \\ 3/7 \end{bmatrix}$$

$$\text{rref}\left(\left[\begin{array}{cc|c} 1 & 4 & 2 \\ 2 & 1 & 1 \end{array} \right] \right) = \left(\begin{array}{cc|c} 1 & 0 & 0.2857 \\ 0 & 1 & 0.4286 \end{array} \right) \checkmark$$

verified

There is one way to check whether an inverse even exists and that is to check a quantity known as the determinant. For now we simply provide the definition for a 2x2 matrix

$$\det\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = a_{11}a_{22} - a_{12}a_{21}$$

if $\det(A) \neq 0$ then A is invertible and the inverse A^{-1} exists!

$\Rightarrow$ unique solution

Try!

$$\det\left(\begin{bmatrix} 1 & 4 \\ 2 & 1 \end{bmatrix}\right) = 1\cdot1 - 4\cdot2 = -7 \checkmark$$

invertible matrix