Lecture 7

Summary: Thus far we have discussed methods of analyzing and solving a system of linear equations of the form

\[ L_1 : a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \]

\[ L_2 : a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \]

\[ \vdots \]

\[ L_m : a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \]

We put this in the augmented matrix form

\[ \begin{pmatrix}
  a_{11} & a_{12} & \ldots & a_{1n} & | & b_1 \\
  a_{21} & a_{22} & \ldots & a_{2n} & | & b_2 \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  a_{m1} & a_{m2} & \ldots & a_{mn} & | & b_m \\
\end{pmatrix} \]

Which allowed us to more readily apply elementary row operations

\[ L_i \leftrightarrow L_j \quad kL_i \rightarrow L_i \quad kL_i + L_j \rightarrow L_j \]

to reduce the system to echelon form

Echelon form reveals many properties of the system including:
- Existence and uniqueness of solutions
- Linear dependence/independence of \( L_1, \ldots, L_m \)
- \# pivot variables and free variables
Continuing to apply elementary operations until the pivot point (first non-zero element in each row) is the only non-zero term in the column gives us row canonical form (reduced row echelon form RREF).

Last lecture we considered the special case where in echelon form the number of equations and unknowns are equal.

Looks like:

\[
\begin{pmatrix}
1 & 0 & \ldots & 0 & \tilde{b}_1 \\
0 & 1 & \ddots & \vdots & \vdots \\
\vdots & \ddots & \ddots & 0 & \vdots \\
0 & \ldots & 0 & 1 & \tilde{b}_n \\
\end{pmatrix}
\]

\(n\) equations, \(n\) unknowns

in row canonical form

Then, it is clear that a unique solution exists.

The matrix \(I\) is the identity matrix.

The identity matrix is such that

\[
\begin{cases}
0 & \text{if } i \neq j \\
1 & \text{if } i = j
\end{cases}
\]

So there are only 1's on the diagonal and the rest are 0's.
Last lecture we introduced the concept of the inverse.

A square matrix \( A \) is said to be invertible or non-singular if there exists a matrix \( B \) such that
\[
AB = BA = I
\]
where \( I \) is the identity matrix such a matrix \( B \) is unique. \( B \) is called the inverse of \( A \) denoted by \( A^{-1} \)

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**Side on matrix algebra**

The product \( AB \) is defined as follows, if \( a_{ij} = [A]_{ij} \)
and \( b_{ij} = [B]_{ij} \) then
\[
[c_{ij}] = [AB]_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}
\]

\[
A \quad B = C
\]
\[(m \times n) (n \times p) = (m \times p)\]
have to be the same

Suppose \( B = [\tilde{b}_1, \tilde{b}_2, \tilde{b}_3, \tilde{b}_4, \ldots, \tilde{b}_p] \quad \tilde{b}_i \in \mathbb{R}^{m \times 1} \)
\[ A\mathbf{b} = [A\mathbf{b}_1, A\mathbf{b}_2, \ldots, A\mathbf{b}_p] \]

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Back to our discussion

What is the role of the inverse?

The inverse contains all the elementary row operations required to get to row canonical form

\[ A^{-1}A = I \]

\[ \text{Inverse identity} \]

Why is this useful? Well if we know \( A^{-1} \) we can solve the system \( A\mathbf{x} = \mathbf{b} \)

\[ A^{-1} (A\mathbf{x} - \mathbf{b}) \]

\[ A^{-1}A\mathbf{x} = A^{-1}\mathbf{b} \]

by definition

\[ I\mathbf{x} = A^{-1}\mathbf{b} \]

\[ \mathbf{x} = A^{-1}\mathbf{b} \]
Goals: Calculate the inverse

One method involves the exact method applied to reach row canonical form

We start with the following augmented matrix

\[
\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} & 1 & 0 & \cdots & 0 \\
a_{21} & \vdots & & \vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn} & 0 & \cdots & 0 & 1
\end{pmatrix}
\]

A note: requires a square matrix

Example

\[
\begin{bmatrix}
1 & 4 & 1 & 0 \\
2 & 1 & 0 & 1
\end{bmatrix}
\]

\[-2L_1 + L_2 \rightarrow L_2\]

\[
\begin{bmatrix}
1 & 4 & 1 & 0 \\
0 & -7 & -2 & 1
\end{bmatrix}
\]

\[-\frac{1}{7} L_2 \rightarrow L_2\]

\[
\begin{bmatrix}
1 & 4 & 1 & 0 \\
0 & 1 & \frac{24}{7} & -\frac{11}{7}
\end{bmatrix}
\]
\[-4L_2 + L_1 \rightarrow L_1\]

\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
-\frac{8}{7} + 1 & \frac{4}{7} \\
\frac{2}{7} & -\frac{1}{7}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]

Let's try it out!

\[
A^{-1}A = \begin{bmatrix}
-1/7 & 4/7 \\
2/7 & -1/7
\end{bmatrix}
\begin{bmatrix}
1 & 4 \\
2 & 1
\end{bmatrix}
\]

\[
A^{-1}[\begin{bmatrix}1 \\ 2\end{bmatrix}] = \begin{bmatrix}1 \\ 0\end{bmatrix}
\]

\[
A^{-1}[\begin{bmatrix}4 \\ 1\end{bmatrix}] = \begin{bmatrix}0 \\ 1\end{bmatrix}
\]

\[
\begin{bmatrix}
-\frac{1}{2} + \frac{9}{7} & -\frac{4}{2} + \frac{9}{7} \\
\frac{2}{7} & -\frac{1}{7}
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\]

yay!
Let's solve the system

\[ A\mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \]

\[ \Rightarrow \mathbf{x} = A^{-1} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \]

\[ = \begin{bmatrix} -1/7 & 4/7 \\ 2/7 & -1/7 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \]

\[ = \begin{bmatrix} -\frac{2}{7} + \frac{4}{7} \\ \frac{4}{7} - \frac{1}{7} \end{bmatrix} = \begin{bmatrix} 2/7 \\ 3/7 \end{bmatrix} \]

\[ \text{ref} \left( \begin{bmatrix} 1 & 4 & 1 \\ 2 & 1 & 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0.2857 \\ 0 & 1 & 0.4286 \end{bmatrix} \]

Verified
There is one way to check whether an inverse even exists and that is to check a quantity known as the determinant. For now we simply provide the definition for a 2x2 matrix

\[
\text{det} \left( \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \right) = a_{11}a_{22} - a_{12}a_{21}
\]

if \( \text{det}(A) \neq 0 \) then \( A \) is invertible and the inverse \( A^{-1} \) exists!

\( \Rightarrow \) unique solution

Try:

\[
\text{det} \left( \begin{bmatrix} 1 & 4 \\ 2 & 1 \end{bmatrix} \right) = 1(1) - 4(2) = -7 \checkmark
\]

invertible matrix