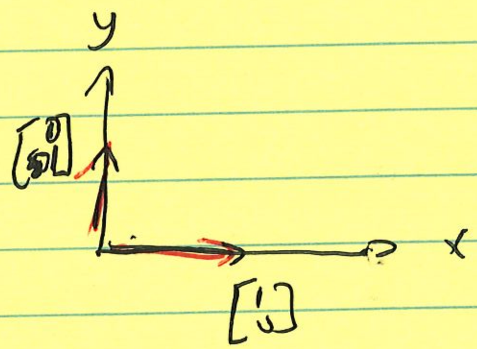
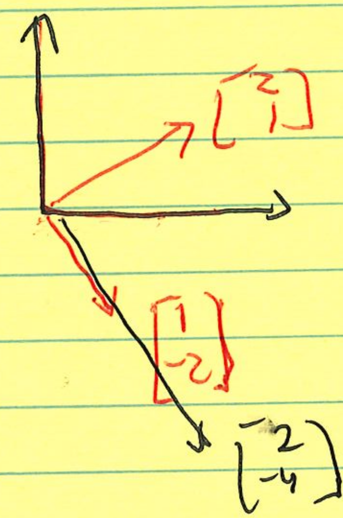
Lecture 7

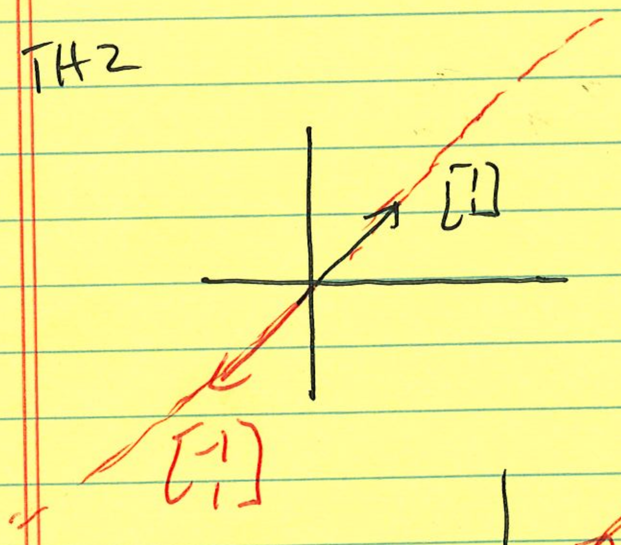
TH1 $\quad \vec{u}, \vec{v}, 5\vec{u} - 3\vec{v}$

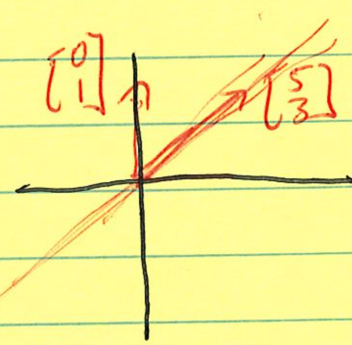
$$\underset{k_1}{-5\vec{u}} + \underset{k_2}{3\vec{v}} + (5\vec{u} - 3\vec{v}) = 0 \qquad k_3 = 1$$

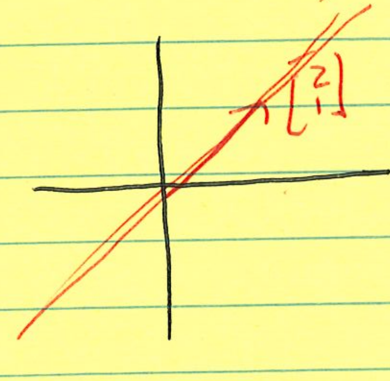




TH2







Let's revisit vector representation for 2D system

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

$$\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} x_1 + \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} x_2 = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\underset{A}{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}} \underset{\vec{x}}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \underset{\vec{b}}{\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}}$$

$A\vec{x} = \vec{b}$ denotes
Matrix vector product

Let's drop $\vec{b}$ for a moment and just think about the Matrix A

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 6 \\ 4 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{\vec{v}} = 1 \cdot \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} + 2 \cdot \begin{bmatrix} 2 \\ 6 \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 5 \\ 15 \\ 4 \end{bmatrix}}_{\vec{u}}$$

$$\underset{A}{(3\times 2)}\ \underset{\vec{v}}{(2\times 1)} = \underset{\vec{u}}{(3\times 1)}$$

$$\begin{bmatrix} 1 & 2 & 8 \\ 3 & 6 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ ? \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 2 \cdot \begin{bmatrix} 2 \\ 6 \end{bmatrix} \quad \times$$

not defined

Properties of $A\vec{x}$

$$A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v}$$

$$A(c\vec{u}) = cA\vec{u} \qquad c \in \mathbb{R}$$

$$A\vec{0} = \vec{0}$$

The identity matrix I $\qquad I \in \mathbb{R}^{3\times 3}$

$$AI = IA = A \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

It's analogous to multiplying by 1!

Example

$$\begin{bmatrix} 3 & 2 & 0 \\ 1 & 0 & 2 \\ -1 & 4 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = 0 \cdot \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} + 1 \cdot \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix} + -1 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$$

Properties of the matrix

For $\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} x_1 + \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} x_2$

$$\Downarrow$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

~~cost~~ $\text{colsp}(A) = \text{Span}\left\{ \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}, \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} \right\}$

column space

$$A\vec{x} = \vec{b} \Leftarrow \vec{b} \in \text{colsp}(A)$$

row space of A denoted rowsp(A)

$$\text{rowsp}(A) = \text{span}\{[a_{11}, a_{12}], [a_{21}\ a_{22}]\}$$

row space and column space are vector subspaces

If $\mathbb{R}^n$ is a vector space and W is a subspace, then the following conditions hold for operations defined in $\mathbb{R}^n$

(a) $\vec{0} \in W$ In $\mathbb{R}^2$ $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

(b) $\vec{x} + \vec{y} \in W$ whenever $\vec{x} \in W$, $\vec{y} \in W$

(c) $a\vec{x} \in W$ whenever $\vec{x} \in W$

Note that $\text{colsp}(A) = \text{rowsp}(A^T)$

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 4 & -1 & 0 \end{bmatrix} \qquad \underset{(A)}{\text{colsp}} = \text{span}\left\{\begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix}\right\}$$

$$A^T = \begin{bmatrix} 1 & 4 \\ 0 & -1 \\ 3 & 0 \end{bmatrix} \qquad \text{rowsp}(A^T) = \text{span}\{[1\ 4], [0\ -1], [3\ 0]\}$$

Basis and dimensions

basis for $\mathbb{R}^2$

$\left\{ \begin{array}{l} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{linearly independent} \\ \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} = \mathbb{R}^2 \end{array} \right]$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ linearly dependent]

$\text{span}\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\} = \mathbb{R}^2$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ linearly independent

$\text{span}\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} = \begin{bmatrix} a \\ b \end{bmatrix}$ s.t. $a = 0$, $b \in \mathbb{R}$]

Definition basis

Def A: A set $S = \{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n\}$ of vectors is a basis of vector space V if it has the following two properties

(1) S linearly independent

(2) S spans V

Def B: A set $S = \{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n\}$ of vectors is a basis of V if every vector $\vec{v} \in V$ can be written <u>uniquely</u> as a linear combination of the basis vector.

$\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ a vector in $\mathbb{R}^2$

For basis $\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot 2 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot 3 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

unique linear combination

For set $\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

example:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot 3 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot 2 + 0 \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$1 \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} + -1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 1 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

infinite possibilities for linear combinations

Example: Suppose you have the following vectors

$\vec{u}_1 = [1, 2, 1, 3]$ $\vec{u}_2 = [1, 3, 3, 5]$

$\vec{u}_3 = [3, 8, 7, 13]$ $\vec{u}_4 = [1, 4, 6, 9]$

$W = \text{span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\}$

Step 1:

$$M = \begin{bmatrix} \vec{u}_1 \\ \vec{u}_2 \\ \vec{u}_3 \\ \vec{u}_4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 1 & 3 & 3 & 5 \\ 3 & 8 & 7 & 13 \\ 1 & 4 & 6 & 9 \end{bmatrix}$$

* construct a matrix where the rows are the vectors

Step 2: Row reduce M to echelon form

$$\mathrm{rref}(M) = \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \tilde{M}$$

$\mathrm{rowsp}(M) = \mathrm{rowsp}(\tilde{M})$

Elementary operations preserved the row space

M and $\tilde{M}$ are <u>row equivalent</u>

In echelon form, remaining rows are linearly independent

Step 3: Output nonzero rows of the echelon matrix

$\vec{v}_1 = [1, 0, 0, 5]$

$\vec{v}_2 = [0, 1, 0, -2]$

$\vec{v}_3 = [0, 0, 1, 2]$

$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for W

$\dim(W) = 3$ (3 vectors in the basis)

$\mathrm{rank}(M) = \mathrm{rank}(\tilde{M}) = \dim(W) = 3$