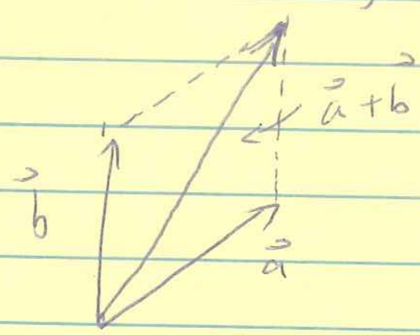
Lecture 8

When we solved $\vec{x}$ in the system $A\vec{x}=\vec{b}$ we were in another sense solving for the linear combination of a set of vectors $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ that gave $\vec{b}$

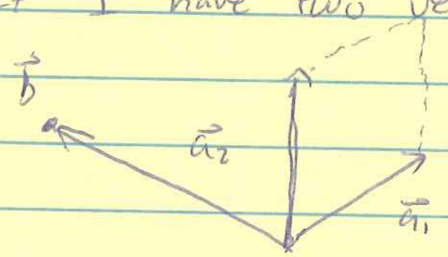
$$\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} x_1 + \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} x_2 + \ldots + \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} x_n = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

In the beginning of the quarter we introduced vector addition by the parallelogram law

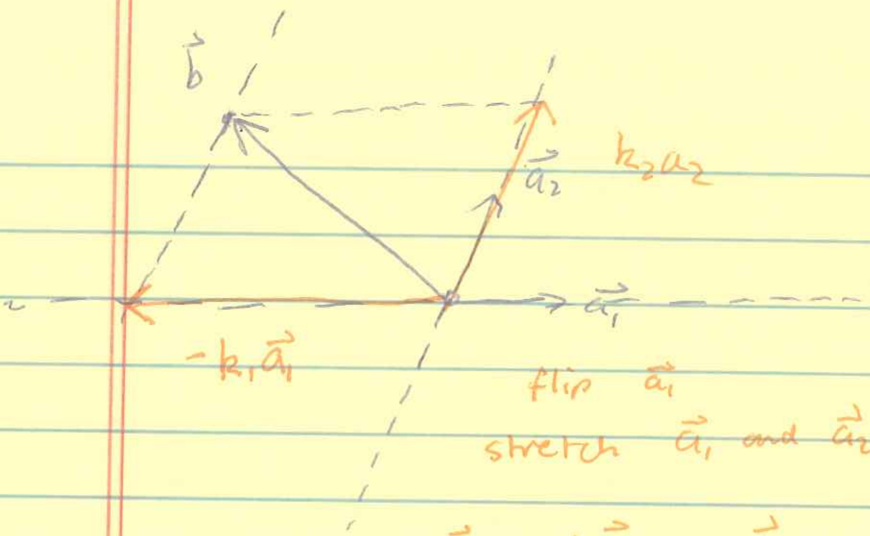



Now suppose I ask the following question:

If I have two vectors $\vec{a}_1, \vec{a}_2 \in \mathbb{R}^{2\times 1}$




how can I scale them and combine them to get my desired vector $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$




$k_1, k_2$ are scaling factors

flip $\vec{a}_1$

stretch $\vec{a}_1$ and $\vec{a}_2$

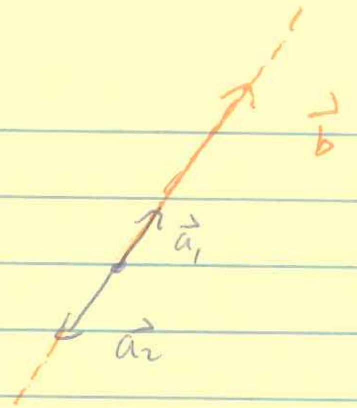
$\vec{a}_1 x_1 + \vec{a}_2 x_2 = \vec{b}$ → solution $x_1 = -k_1$, $x_2 = k_2$

$$\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} x_1 + \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} x_2 = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$-k_1 \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + k_2 \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

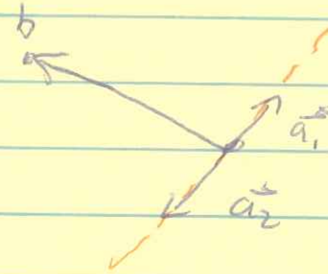
Suppose our two vectors $\vec{a}_1$ and $\vec{a}_2$ where in line with each other


Note that I can only find a solution to $\vec{a}_1 x_1 + \vec{a}_2 x_2 = \vec{b}$ if $\vec{b}$ is in line with $\vec{a}_1$ and $\vec{a}_2$, then we have infinite solutions




If $\vec{b}$ along the dashed line we have infinite ways to combine $k_1\vec{a}_1$ and $k_2\vec{a}_2$ to get $\vec{b}$

OR




No solution is possible!

Example

$A = [\vec{a}_1, \vec{a}_2]$ $\quad \vec{a}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \vec{a}_2 = \begin{bmatrix} -2 \\ -4 \end{bmatrix}$

$A\vec{x} = \vec{b}$

Note that this is the situation where

$$x_1 - 2x_2 = b_1$$
$$2x_1 - 4x_2 = b_2$$

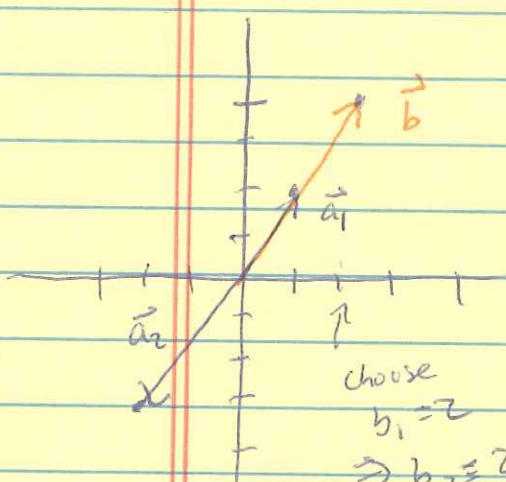
$$-2L_1 + L_2 \to L_2$$

$$x_1 - 2x_2 = b_1$$
$$0 + 0 = -2b_1 + b_2$$

if $-2b_1 + b_2 = 0$, then the system is consistent

$$\Rightarrow b_2 = 2b_1$$

Choose $b_1$ and $b_2$ is twice that




choose $b_1 = 2$

$\Rightarrow b_2 = 2b_1 = 4$

So far, we have been asking how can we combine vectors to get a specific answer

i.e. solve $A\vec{x} = \vec{b}$ find $\vec{x}$ that gives $\vec{b}$

Another important subject of linear algebra revolves around the question

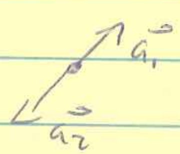
If I have a set of vectors $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ what are the possible $\vec{b}$'s I can get.

In the first case


$\vec{a}_1$ and $\vec{a}_2$ can be linearly combined to produce any other vector in 2-D space

We say $\vec{a}_1$ and $\vec{a}_2$ generates (or spans) the entire 2-dimensional space $\mathbb{R}^{2\times 1}$ (all $2\times 1$ vectors)

In the second case




$\vec{a}_1$ and $\vec{a}_2$ generate or span a subspace of $\mathbb{R}^{2\times 1}$ but not the complete vector space $V$

Subspace is generated by

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

it is also generated by

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

or

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

but the last two examples are linearly dependent sets and so a larger set doesn't change the space of solutions so $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is sufficent

Theorem 1.3 Let $V$ be a vector space and $W$ be a subset of $V$. Then $W$ is a subspace of $V$ if and only if the following conditions hold for the operations defined in $V$

(a) $0 \in W$

(b) $x + y \in W$ whenever $x \in W$, $y \in W$

(c) $ax \in W$ whenever $x \in W$

That means if $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ generate a subspace then

(a) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ must also be a part of space $W$

(b) $\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$ is also in $W$

2:1 ratio is maintained

(c) $ax \in W$

$$5 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$$ also is in $W$

same 2:1 ratio

If we apply any of the operations shown using vectors in the ~~sp~~ subspace $W$, then the answer stays in $W$

Hence, it is sufficient to define the subspace $W$ in our example by the single vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

What about our 1st example when $\vec{a}_1$ and $\vec{a}_2$ were <u>not</u> parallel?

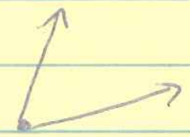
Well we need both to generate any $2 \times 1$ vector so they are both necessary, and linearly indendent.

Definition: We say that vectors $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ in $V$ are linearly <u>dependent</u> if there exist scalars $k_1, \ldots, k_n$, not all zero, such that

$$k_1 \vec{a}_1 + k_2 \vec{a}_2 + \cdots + k_n \vec{a}_n = 0$$

otherwise, we say that they are linearly independent

$\vec{a}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$



We claim they linearly ~~indendent~~ independent

$\vec{a}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Question

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If they are linearly independent then there does not exist $k_1 \neq 0$ or $k_2 \neq 0$ s.t. the above equation is satisfied.

$\Rightarrow$ The only solution is $k_1 = 0$ and $k_2 = 0$

$$2k_1 + k_2 = 0$$
$$k_1 + k_2 = 0$$

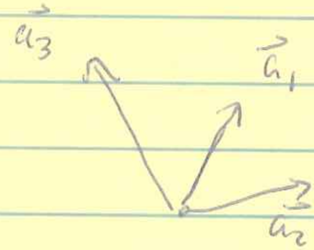
$$-\frac{1}{2}L_1 + L_2 \rightarrow L_2$$

$$2k_1 + k_2 = 0$$
$$0 + \frac{1}{2}k_2 = 0 \qquad \Rightarrow k_2 = 0$$

$$k_1 = -\frac{1}{2}k_2$$
$$= 0 \checkmark$$

Yes, they are linearly independent

Note that adding a third vector to the set makes it linearly dependent




$\vec{a}_1, \vec{a}_2, \vec{a}_3$ is a linearly dependent set

$\vec{a}_3$ can be constructed from $\vec{a}_1$ and $\vec{a}_2$